Functorial affinization of Nash's manifold

Abstract. Let M be a singular irreducible complex manifold of dimension n. There are $\mathbb{Q}$ divisors $D[-1], D[0], D[1], ..., D[n+1]$ on Nash's manifold $U \to M$ such that $D[n+1]$ is relatively ample on bounded sets, $D[n]$ is relatively eventually basepoint free on bounded sets, and $D[-1]$ is canonical with the same relative plurigenera as a resolution of M. The divisor $D = D[n]$ is the supremum of divisors $\frac{1}{i}D_i$. An arc containing one singular point of M lifts to U if and only if the generating number of $\oplus_i \mathcal{O}_\gamma(D_i)$ is finite. When finite it equals $1 + (K_U - K) \cdot \gamma$ where $\mathcal{O}_U(K)$ is the pullback mod torsion of $\Lambda^n \Omega_M$. If C is a complete curve in U then $\frac{-1}{n+1} K_U \cdot C = D_1 \cdot C + D_{n+2} \cdot C + D_{(n+2)^2} \cdot C + ...$ When there are infinitely many nonzero terms the sum should be taken formally or p-adically for a prime divisor p of $n+2$. There are finitely many nonzero terms if and only if $C \cdot D = 0$. The natural holomorphic map $U \to M$ factorizes through the contracting map $U \to Y_0$. The Grauert-Riemenschneider sheaf of M, if M is bounded, agrees with $\mathcal{H}om(\mathcal{O}_M(D_{(n+2)^i-1}),\ \mathcal{O}_M(D_{(n+2)^i}))$ for large i. If M is projective, singular s-dimensional foliations on M such that $K + (s+1)H$ is a finitely-generated divisor of Iitaka dimension one are completely resolvable, where K is the canonical divisor of the foliation.

According to a question of [7] it is not known whether Y_0 always has canonical singularities. It is not known whether the analogue of Nash transforms, locally principalizing the reflexivication of $\Lambda^n \Omega_M$, eventually converges, but this is true for toric surfaces. Conjecturally a split faithful action of a commutative Lie algebra can never be completely resolved if the weights do not form a basis (possibly with multiplicity) of the dual of the Lie algebra. This is not known. It is not understood under what conditions Y_0 can be connected to the relative canonical model by proper maps over M.

John Atwell Moody
Coventry
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0. Preliminaries

0.1 Hypotheses. Throughout this article, M will be a singular complex manifold; that is, a Hausdorff space with a countable open cover by closed analytic subspaces of complex domains, furnished with the reduced structure sheaf. We suppose that M is irreducible and let $n = dim(M)$. An open subset of M will be called *bounded* if its closure is compact.

0.2 The Nash manifold. Let $...M_2 \to M_1 \to M_0 = M$ be the sequence of Nash blowups of M [4] and let $U_0 \subset U_1 \subset U_2 \subset ...$ be the ascending chain of open immersions, where U_i is the regular locus of M_i. Let U be the colimit, so U is the ascending union of the U_i; every point of U is contained in one of the U_i and the U_i are complex manifolds; and so U is also a complex manifold. It is, and is here also defined to be, Nash's manifold. It is not known whether the structural map $U \to M$ is always proper, and this is a question raised by Nash's work.

0.3 Canonical Singularities. For each natural number r let $V_r \to M$ be the universal singular manifold such that the push-forward of $\omega^{\otimes r}$ from a resolution of V_r is invertible and relatively basepoint free over M. The V_i have inclusions according to divisibility

$$\begin{array}{ccccc} & & V_2 & & \\ & \nearrow & & \searrow & \\ V_1 & & & & V_6 \\ & \searrow & & \nearrow & \\ & & V_3 & & \end{array}$$

etc. Set V to be the colimit of the V_i. The maps $V_i \to V$ are open immersions providing an open cover. Since each V_i has canonical singularities, V itself is a manifold with canonical singularities. That is to say, if $s : V' \to V$ is a resolution then every point of V has an index r and a Stein neighbourhood T such that the $\Gamma(s^{-1}T, \omega^{\otimes r})$ is a principal $\Gamma(T, \mathcal{O}_V)$ module. As in the case of the Nash manifold, there is a natural map $V \to M$, conjectured to be a proper map [7], now known when the singularities of M are isolated, and much more generally ([13,21,22,23] further discussion in the appendix).

0.4 Constructions of Grothendieck and Atiyah

Let now $\mathcal{F}$ be any coherent analytic sheaf on M, we assume $\mathcal{F}$ is torsion-free and of rank one. Let $f : \mathbf{V}(\mathcal{F}) \rightarrow M$ be Grothendieck's map of singular manifolds [2], definition 1.7.8, with analytic action of $Gl_1(\mathbb{C})$ freely and transitively on the fibers with fixed locus M itself, which is characterised by the property that for $i \in \mathbb{Z}$, the coherent sheaf of complex valued holomorphic functions on F which transform according to the character of degree i are just 0 if $i < 0$ and the power $\mathcal{F}^{\otimes i}/torsion$ otherwise. If $\mathcal{F}$ is invertible, then $\mathbf{V}(\mathcal{F})$ can be identified point-by-point with the dual of the line bundle whose section sheaf is $\mathcal{F}$. In general, [2] paragraph 1.7.9, the section sheaf of $\mathbf{V}(\mathcal{F})$ is $\mathcal{H}om(\mathcal{F}, \mathcal{O}_M)$. The subsheaf of $f_*\Omega_{\mathbf{V}(\mathcal{F})}/torsion$ which transform according to the character of degree one is called the *principal parts sheaf of $\mathcal{F}$ modulo torsion* and will be denoted $\mathcal{P}(\mathcal{F})$, and the exact sequence $0 \rightarrow \mathcal{F} \otimes \Omega_M/torsion \rightarrow \mathcal{P}(\mathcal{F}) \rightarrow \mathcal{F} \rightarrow 0$ is Atiyah's sequence [1]. There is a natural splitting i of Atiyah's sequence as a sheaf of complex vector-spaces, and the sheaf of $\mathcal{O}_M$-module splittings assigns to an open set $T \subset M$ precisely the $i + \nabla$ where $\nabla : \mathcal{F}_{|T} \rightarrow \mathcal{P}(\mathcal{F}_{|T})$ runs over the distinct connections on the restriction of $\mathcal{F}$ to T.

1. The ordering on coherent sheaves

1. Theorem (see [17]) If $\mathcal{F}$ and $\mathcal{G}$ are two such coherent sheaves then there is a natural transformation

$$\mathcal{G}^{n+1}\Lambda^{n+1}\mathcal{P}(\mathcal{F}) \to \Lambda^{n+1}\mathcal{P}(\mathcal{G}\mathcal{F}) \quad (1)$$

such that for each $\mathcal{F}$ and $\mathcal{G}$, the map pulls back on $Bl_{\mathcal{F}\mathcal{G}}(M)$ to the natural map (where $\tau : Bl_{\mathcal{G}\mathcal{F}}M \to Bl_{\mathcal{F}}M$ over M)

$$\tau^*\Lambda^n\Omega_{Bl_{\mathcal{F}}M}/torsion \to \Lambda^n\Omega_{Bl_{\mathcal{F}\mathcal{G}}M}/torsion$$

twisted by the $n+1$ power of the invertible sheaf which results by pulling back $\mathcal{F}\mathcal{G}$ and reducing mod torsion. The map is not an $n+1$ exterior power. It generalizes the special case when $Bl_{\mathcal{F}}M$ is normal.

2. Corollary. Suppose $Bl_{\mathcal{F}}M$ is normal. Then, on each bounded open subset of M, $\mathcal{G}$ is a divisor of a power of $\mathcal{F}$ as sheaves of fractional ideals if and only if (1)/torsion becomes surjective after multiplying by a power of $\mathcal{F}\mathcal{G}$.

Among multiplicatively closed 'sets' of torsion-free coherent sheaves on M closed under division and multiplication, those which happen to be finitely generated on bounded open sets of M , are generated by a single element on each bounded open set. Such multiplicatively closed sets of sheaves correspond bi-uniquely with singular manifolds $N \to M$ over M whose structure map is locally projective of degree one. The corollary for example can be interpreted as saying that on bounded open sets the map (1) is an isomorphism after multiplying by some power of $\mathcal{F}\mathcal{G}$ if and only if $\mathcal{G}$ is already contained in the smallest multiplicatively and divisbilitively closed set containing $\mathcal{F}$.

Stein factorization implies that finitely generated multiplicatively and divisibilitively closed sets of torsion free coherent sheaves of rank one have the descending chain property on bounded open subsets of M. The unique minimal set for example is the set of invertible sheaves. The corresponding chain condition can be included in the conclusion of the theorem.

2. A basepoint free theorem

Let $f : N \to M$ be a locally projective morphism of degree one. Generalizing from the case of invertible sheaves, we say a torsion-free coherent sheaf of rank one $\mathcal{F}$ on N is *spanned relative to* f if each point p of M has a neighbourhood such that the restriction of $\mathcal{F}$ to the inverse image of the neighbourhood is generated by global sections; and we say that $\mathcal{F}$ is *very ample relative to* f if all $\mathcal{F}^{\otimes i}/torsion$ are spanned relative to f and the meromorphic map $N - \to \mathcal{P}roj \oplus_i (f_*\mathcal{F})^i$ is the inverse of a morphism.

3. Corollary. Let $\mathcal{L}$ now be an invertible sheaf on N which is very ample relative to f. Then $\mathcal{L}^{n+1}\Lambda^n\Omega_N/torsion$ is spanned relative to f and $\mathcal{L}^{n+2}\Lambda^n\Omega_N/torsion$ is very ample relative to f.

The inverse morphism is the Nash blowup, which makes its appearance in this way.

The primary difference between Miles' model of [7] and the Nash model is that Miles' model has a functorial and canonical affinization. This difference disappears upon contemplation of the formula for the partial sum of a geometric series

$$(n+2)^s = (n+1)(1 + (n+2) + ... + (n+2)^{s-1}).$$

Namely, starting from a torsion-free coherent sheaf $\mathcal{F}$ on M, we define for each natural number i a new torsion free coherent sheaf, functorial with respect to Grothendiecks' category of holomorphic maps of M and coherent sheaf maps of $\mathcal{F}$, by the rule that if the base-$(n+2)$-expansion of i is $a_0 + a_1(n+2) + ... + a_s(n+2)^s$ with $0 \leq a_i < (n+2)$, then we write $\mathcal{F}_i = \mathcal{F}_1^{\otimes a_0} \otimes ... \otimes \mathcal{F}_{(n+2)^s}^{\otimes a_s}/torsion$ and when $i = (n+2)^j$ is a power of $n+2$ we write $\mathcal{F}_i = \Lambda^{n+1}\mathcal{P}(\mathcal{F} \otimes \mathcal{F}_{\frac{i-1}{n+1}})/torsion$. The morphism of theorem 1. provides the needed 'carrying' map

$$\mathcal{F}_{(n+2)^j}^{\otimes(n+2)} \to \mathcal{F}_{(n+2)^{j+1}}$$

such that the $\mathcal{F}_i$ can be multiplied following the usual manner in integer expressions to the base $n+2$ are added together.

4. Theorem. If $\mathcal{F}$ is a very ample invertible sheaf on M then the $\mathcal{F}_i$ are torsion free coherent sheaves on M which are generated by global sections.

3. The next-to last stage of Nash's tower

5. Theorem. Suppose the Nash tower $...M_{m+1} \to M_m \to ... \to M_0 = M$ satisfies $M_{m+1} = M_m$. Then $U = M_m$, $\oplus_i \mathcal{F}_i$ is finite type (although the local generating degree can be larger than $(n+2)^m$ and might not be bounded) and taking $Y = \mathcal{P}roj \oplus \mathcal{F}_i$ there is a pullback diagram

$$\begin{array}{ccc} M_m & \to & M_{m-1} \\ \downarrow & & \downarrow \\ Y & \to & M \end{array} .$$

In other words, $U = M_m$ can always be built by pulling back the penultimate term of the Nash tower $M_{m-1} \to M$ along $Y \to M.$ In general, by section 6, properness of $U \to Y$ is equivalent to $\oplus \mathcal{F}_i$ being (locally) of finite type.

Proof. $\mathcal{F}_{(n+2)^m-1} = \mathcal{F}_{(n+2)^{m-1}-1} \otimes \mathcal{F}^{n+1}_{(n+2)^{m-1}} / torsion$ while M_m, M_{m-1}, Y are the blowups of M along the three sheaves in the equation.

4. Properness of $U \to Y_0$

If it is not known whether $U \to Y$ is proper, instead let $f : Y_0 \to M$ be the universal singular manifold over M such that at every point $p \in Y_0$ there is an index r such that $f^* \mathcal{F}_i / torsion$ is principal in a neighbourhood of p when $r|i.$

6. Corollary. The map $U \to Y_0$ is always proper. Therefore $U \to M$ is proper if and only if $Y_0 \to M$ is proper.

Proof. The equation above shows that the local isomorphism type of the pullback modulo torsion of $\mathcal{F}_{(n+2)^m-1}$ stabilizes as m increases. This does immediately imply the corollary without further work, but a less mysterious proof is by considering arc lifting as we shall do in section 8.

5. The description of U as a graph

7. Theorem. Suppose M is projective with very ample $\mathcal{F}$ and that $Y_0 \to M$ is proper. Let $W = Proj \oplus_i \Gamma(\mathcal{F}_i)$. Then W is an algebraic variety and $Y_0 \to M$ is not only a proper map, it is actualy the universal solution of resolving the indeterminacies of the rational map $M - \to W$, ie it is the closure of the 'graph' of the rational map $M - \to W$.

The following is an immediate consequence of theorems 5. and 7.

8. Corollary. The map $M - \to W$ is a morphism if and only if M is nonsingular.

9. Remark. Assume that M is a singular projective variety. Then the singular manifold W consists of a single point if and only if M is a linear projective space and H is a hyperplane.

This shows that $dim(W)$ need not *always* be as large as n. It is of course bounded above by the Iitaka dimension of $K + (n+1)H$. We have to be careful if we want to work by induction on Iitaka dimenson as for any Gorenstein variety besides projective space which has isolated irrational singularities, $K + (n+1)H$ is very ample (see [16] theorem 8.8.5). Beginning in section 7 we'll consider the consequences of forcing the calculation when n is intentionally chosen to be different than the dimension of M.

6. The terms of degree $\frac{(n+2)^i-1}{n+1}$.

The terms $\mathcal{F}_i$ where i is a partial sum of a geometric series to the base of $n+2$ play a particular role, as if I replace $\mathcal{F}$ by such an $\mathcal{F}_i$ then the auxiliary singular manifold Y_0 is unaffected, and the effect on the sequence $\mathcal{F}_0, \mathcal{F}_1, ...$ is to 'truncate' by passing to the subsequence of multiples of a power $(n+2)^j$.

Properness of $Y_0 \to M$ is equivalent $\oplus \mathcal{F}_i$, being (=locally) of finite type. That is to say, on each bounded open subset of M $\mathcal{F}_{(n+2)^i}^{\otimes(n+2)} \to \mathcal{F}_{(n+2)^{i+2}}$ is a surjective map of rank one coherent sheaves for large i. Once this happens for one value of i then it happens for all sufficiently large values of i. Surjectivity of the map from the $n+2$ symmetric power of the terms of degree 1 to terms of degree $n+2$ is neither necessary nor sufficient for nonsingularity of $Bl_{\mathcal{F}}(M)$, but it follows nevertheless therefore from Hironaka's theorem [3] and [20] Theorem 3.45 (see also crucial references therein) that on a bounded open set once $\mathcal{F}$ is replaced by such an $\mathcal{F}_i$ that there is always a choice of $\mathcal{F}$ such that $\oplus_i \mathcal{F}_i$ has generators of degree one. Conversely when $\mathcal{F}_1$ generates, then $\mathcal{F} \otimes \mathcal{F}_1/torsion$ is a resolving sheaf.

7. Filtration of the homogeneous coordinate ring of V

The results above provide a functorial relative affinization of Nash's manifold which is analogous to the intrinsic relative affinization of $V.$ This is not precisely automatic, for example no functorial affinization of Hironaka's and Spivakovsky's model [14] is known.

If n is taken to be $1 + dim(M)$ in the definition of the multiplication (but not in the exterior degrees) then a series of subsheaves of the $\mathcal{F}_i$ arises. It is confusing to say 'let us no longer assume that n is equal to the dimension of M' and so let's instead continue to let $n = dim(M)$ and use a contrivance: for each integer $N \geq -1,$ whenever j is a power $j = (N+2)^i$ let

$$X_j[N] = \mathcal{F}^{\otimes(n+1)\frac{(N+2)^i-(n+2)^i}{N-n}} \otimes \mathcal{F}_{(n+2)^{i-1}} \otimes \mathcal{F}^{\otimes N+2}_{(n+2)^{i-2}} \otimes \mathcal{F}^{\otimes(N+2)^2}_{(n+2)^{i-3}} \otimes ... \otimes \mathcal{F}_1^{\otimes(N+2)^{i-1}}/torsion$$

depending on N, n and j. The exponent of $\mathcal{F}$ is positive regardless of the relative magnitude of N and $n.$ If j is not a power of $(N+2)$ then define X_j to be a product according to the base $N+2$ expansion of $j,$ and let

$$\mathcal{F}_j[N] = lim_s \mathcal{H}om(X_j[N]^{n+s}, X_j[N]^{N+s} \otimes \mathcal{F}_k/torsion)$$

where k results by replacing N by n in the base $N+2$ expansion of $j.$ (When $N = -1$ and base $N+2$ expansions are ambiguous also take the colimit over expansions). Taking $N = n$ we have

$$\mathcal{F}_j \subset \mathcal{F}_j[n]$$

an integral map. For N taking values besides n then the $\mathcal{F}_j$ are nested

$$\mathcal{F}_j[n+1] \subset \mathcal{F}_j[n] \subset ... \subset \mathcal{F}_j[-1]$$

and it is also easy to see that for each fixed n the $\oplus_j \mathcal{F}_j[N]$ form a graded ring sheaf. We will also abbreviate $\oplus_i \mathcal{F}_i[N]$ by the symbol $\mathcal{F}[N].$

10. Theorem. As N passes along the sequence $-1, 0, 1, 2, ..., n+1$ the $\mathcal{F}[N]$ describe then a filtration of the sheaf of homogeneous coordinate rings of $V.$

Proof. By 14 d) $\mathcal{F}_j[-1] \subset \tau_* \mathcal{O}_U(jK_U).$

For each N let $f : Y_0[N] \to M$ be the universal singular manifold such that each point p has an index r such that $f^*\mathcal{F}_i[N]$ is principal in a neighbourhood of p for all $r|i$. The resulting $Y_0[N]$ is independent of the choice of $\mathcal{F}$ if $\mathcal{F}$ is invertible, and let us take $\mathcal{F} = \mathcal{O}_M$. It will follow from Theorem 14

11. Theorem.

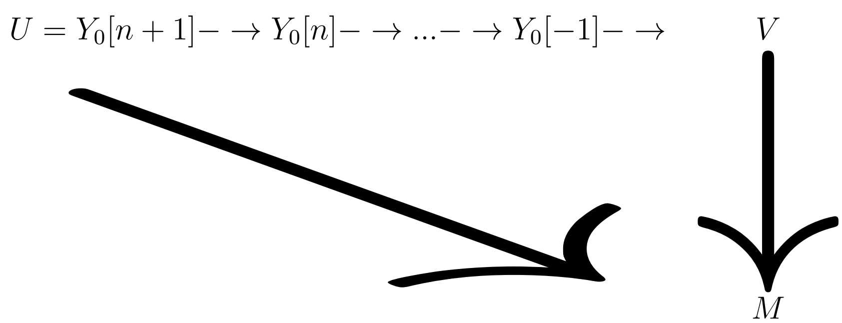
a) When $N = -1$, the integral closure of $\mathcal{F}[N]$ is the homogeneous coordinate ring of V, and so the $Y_0[-1] - \to V$ is the inverse of a finite (=proper locally affine) morphism.

b) When $N = n$, $Y_0[N]$ admits a finite map to Y_0 over M.

c) When $N \geq n+1$, $Y_0[N]$ is isomorphic to U.

d) For values of N between -1 and $n+1$ there are meromorphic maps whose inverses are carried by morphisms of the affinizations.

e) The first map $Y_0[n{+}1] - \to Y_0[n]$ is proper and holomorphic.



It holds from [13,21,22,23] that the map $V \to M$ is a proper morphism if M is locally algebraic (this is full generality if the singularities of M are isolated [5]). The diagonal vertical arrow and the vertical arrow are the two models U and V. The existence of a chain of proper maps in either direction between U and V over M is not a new open question. Unless both were false, it is equivalent to the logical equivalence between the questions considered separately of the properness of U over M and the properness of V over M.

Pulling back $\oplus_j \mathcal{F}_j$ modulo torsion along an analytic arc $t \mapsto \gamma(t) \in M$ which is defined on a domain of times $T \subset \mathbb{C}$ gives a graded ring of coherent sheaves on a domain in the complex number line. Suppose that the arc contains a regular point. Then the sheaf is finitely generated except at a discrete set of times $t,$ so there is no harm assuming that $\gamma(t)$ is a regular (smooth) point except when $t = 0.$ The restriction of $\gamma^* \mathcal{F}^{-n-2}_{(n+2)^i} \cdot \gamma^* \mathcal{F}_{(n+2)^{i+1}}$ to the open set $T \setminus \{0\}$ is canonically isomorphic to $\mathcal{O}_{T\setminus\{0\}}$ and contains a canonical copy to $\mathcal{O}_T$. On T there is a thereby distinguished natural isomorphism of the torsion free part to $t^{-d_i}\mathcal{O}_T$ for a natural number d_i depending on i. Thus we obtain a sequence of natural numbers $d_1, d_2, ...$

12. Theorem. The sum $\sum_{i=1}^{\infty} d_i$ is finite if and only if γ lifts to U, and it is then equal to the local intersection product $(K_U - K)\cdot\gamma$ where K is a Cartier divisor such that $\mathcal{O}_U(K)$ is the pullback mod torsion of $\Lambda^n\Omega_M$. In other words, passing to the local analytic ring $\mathbb{C}\{\{t\}\}$ at the origin, the minimum number of generators $gen(\gamma)$ of the local algebra $\gamma^*(\oplus_i\mathcal{F}_i)_{\{0\}}$ over $\mathbb{C}\{\{t\}\}$ is given by the equation

$$\boxed{gen(\gamma) = (K_U - K)\cdot\gamma + 1}$$

Proof. The integral extension from section 10 $\oplus_i\mathcal{F}_i \subset \oplus_i\mathcal{O}(D_i)$ pulls back modulo torsion to an isomorphism. Then $d_0 = 1$ and $d_i = \gamma \cdot (D_{(n+2)^i} - (n+2)D_{(n+2)^{i-1}})$ for $i \geq 1$. By the definition of the D_j this equals $\gamma \cdot (K_i - K_{i-1})$. Adding over $i = 1, 2, ...$ yields a collapsing sum adding to $\gamma \cdot (K_U - K)$.

For example if C crosses $K_U - K$ transversely at one point then $\gamma^* \oplus \mathcal{F}_i$ is isomorphic to a graded sheaf of algebras generated by t in degree 1 and $t^{(n+2)^s-1}$ in degree $(n+2)^s$ where s is, I believe, the number of Nash blowups needed to make the intersection transverse. The intersection number is then the local winding number of $T \setminus \{0\}$ about one of the components of $K_U - K$.

9. Examples and discussion.

This section will have no theorems, but is included for context. In the case when $\mathcal{F}$ is very ample the $\mathcal{F}[N]$ are generated by global sections as long as $N \geq n$. The square brackets instead of round brackets around N are to avoid confusion with the notation of level in modular forms. For example if M is a modular Riemann surface of level $G \subset \mathbb{P}Sl_2(\mathbb{Z})$ a torsion free group, and $\mathcal{F}$ is the section sheaf of a line bundle, including one global section which crosses each cusp transversely, then in the inclusions of four graded rings

$$\Gamma\mathcal{F}[2] \subset \Gamma\mathcal{F}[1] \subset \Gamma\mathcal{F}[0] \subset \Gamma\mathcal{F}[-1]$$

all four rings are the same; they are all equal to the ring of modular forms of even weight and level G. The composite of the four equalities induces the composite of the Nash resolution with the inverse of the relative canonical morphism. An example where N matters is the case of something higher-dimensional such as the example of Kollar and Ishii [18], the solution set M of the equation $x_1^3 + ... + x_4^3 + x_5^6 = 0$, and let us here repeat the beginning part of their discussion. Here $V \to M$ is an isomorphism because M already has canonical singularities. These can be resolved by blowing up reduced points. First blowing up the singular point yields as exceptional component the projectivized tangent cone at the origin, a cone on a cubic hypersurface in $\mathbb{P}^3$, which is ruled. Nevertheless it must occur in any resolution because of Nash's argument [15] about irreducible components in the space of arcs. The blow-up in turn of the cone point of the projective subvariety provides a resolution $V' \to V$ with a second component, a full cubic hypersurface in $\mathbb{P}^4$ which is not an arc component because first infinitesimal neighbourhoods of lines split; but is now essential because it is non-generically ruled. Thus the paper produces two essential components which live 'above' the smallest canonical singularities model V but for different reasons. Also [18] says that even if we did not now that the first component were an arc component, and even though it is not one of the essential crepant components of [7], its essentialness also follows because of having minimal discrepancy.

In this case $\Lambda^4\Omega_V/torsion = \frac{1}{x_5^5} I dx_1 dx_2 dx_3 dx_4$ has invertible reflexivication $\frac{1}{x_5^5}\mathcal{O}_V dx_1 dx_2 dx_3 dx_4$ where $I = (x_1^2, x_2^2, ..., x_5^5)$. The section $\frac{1}{x_5^5} dx_1 ... dx_4$ of the canonical line bundle of the resolution a has a simple zero at the first exceptional component and a double zero at the second one. Taking $\mathcal{F}$ to be the unit ideal, each $\mathcal{F}_i$ contains $I^i \frac{1}{x_5^{5i}}(dx_1...dx_4)^i$ and is of the form $I_i \frac{1}{x_5^{5i}}(dx_1...dx_4)^i$ for I_i a suitable adjunction ideal. Defining the sequence $g_1, ..., g_5 = x_1^2, ..., x_5^5$, then $\mathcal{F}_1 = I \frac{1}{x_5^5} dx_1 ... dx_4$ is ordinary $4-$ forms, and $\mathcal{F}_6$ contains $\mathcal{F}_1^6$ but is slightly larger; it is spanned over $\mathcal{O}_V$ by the alternating forms

$$g_0 dg_1 dg_2 dg_3 dg_4 - g_1 dg_0 dg_2 dg_3 dg_4 + g_2 dg_0 dg_1 dg_3 dg_4$$

$$-g_3 dg_0 dg_1 dg_2 dg_4 + g_4 dg_0 dg_1 dg_2 dg_3$$

for $g \in \mathcal{F}\mathcal{F}_1 = \mathcal{F}_1$. In other words symmetric linear six-forms in alternating differential four-forms are a special case of an alternating sum of products of differential four-forms against four-forms in four-forms. Amusingly, the expression above can be rewritten as $g_0^5 d(g_1/g_0)...d(g_4/g_0)$ showing that this expression is then antisymmetric in the g_i. This same expression which describes a typical n-form on a coordinate chart with an extra zero of degree $n+1$ on the exceptional locus describes also a typical element in the image of the universal connection. Let us explain this and also something more general.

13. Observation The expression $g_0^{n+1} d(g_1/g_0)...d(g_n/g_0)$ is antisymmetric under permutations of $g_0, ..., g_n$ and results from applying the universal $\mathbb{C}$-linear connection ∇ to $dg_0...dg_n$ using Leibniz rule. Therefore $\mathcal{F}_6$ is generated by the image of ∇ if $g_0, ..., g_4$ run over sections of the product sheaf $\mathcal{F}\mathcal{F}_1$.

Just generally, if $\mathcal{G}$ is torsion-free coherent of rank s then

$$0 \xrightarrow{H} \Lambda^{s(n+1)}\mathcal{P}\mathcal{G} \otimes S^j(\mathcal{G})/torsion \xrightarrow{H} \Lambda^{s(n+1)-1}\mathcal{P}\mathcal{G} \otimes S^{j+1}(\mathcal{G})/torsion \xrightarrow{H} ...$$

an exact complex for $j \in \mathbb{N}$.

The sheaf $\mathcal{P}(\mathcal{G})$ is the kernel of $q : \Omega_{V(\mathcal{G})} \to \Omega_M$, (that is, one-forms which are zero on M) pulled back as a coherent sheaf along the inclusion $i : M \subset V(\mathcal{G})$ of the zero section, to arrive at $i^*Kernel(q)/torsion.$ If y is a section of $\mathcal{G}$ viewed as a linear function on the fibers of f, it is already in $Kernel(q)$, and for x a local holomorphic function on M, the connection ∇ is merely the deRham extending the deRham differential d on M and satisfying $\nabla(xy) = x\nabla(y) + ydx.$

Now pulling back $\mathcal{P}(\mathcal{G})$ further along $f : V(\mathcal{F}) \to M$ gives $f^*\mathcal{P}(\mathcal{G})$, and H and ∇ extend by Leibniz rule. H is a contracting homotopy; the formulas

$$H \circ \nabla + \nabla \circ H = (s(n+1) - j)$$

$$H \circ H = 0$$

$$\nabla \circ \nabla = 0.$$

H commutes with x and sends $dy \mapsto y \mapsto 0.$

The result $f^*\mathcal{P}(\mathcal{G})/torsion$ is the direct sum of the sequences shown above plus a finite number of incomplete parts of sequences exact except possibly at the leftmost term, which is $\Lambda^i(\mathcal{G} \otimes_{\mathcal{O}_M} \Omega_M)$ when i is small enough that this is nonzero. The Atiyah sequence $H : \Lambda^1\mathcal{P}(\mathcal{G}) \to S^1\mathcal{G} \to 0$ with its $\mathbb{C}$-linear splitting $\nabla : \mathcal{G} \to \mathcal{P}(G)$ is one of those incomplete parts, the one for $i = 1$, although the information there determines the other sequences. The Atiyah sequence itself does not transform well since the kernel term $\Omega \otimes \mathcal{G}/torsion$ is nonvanishing on the zero section M. The notion of Atiyah is that although this sequence is exact, still K-theoretic information can be extracted as Whitehead torsion. The same should be true of the exact sequences which transform better too. That is, the exact sequences of sheaves whose sections vanish on the zero section M are not affected by various modification of the zero section such as intermediate blowups between M and $Bl_{\mathcal{G}}(M)$. Though algebraic cycles representing the Whitehead torsion might change dimension reminiscent of dimension changes in arc components.

Esnault, Viehweg and Verdier show in [12], Appendix B, how to extract higher Chern classes using Deligne's concept of connections with logarithmic poles; and they mention that the assumption of a normal crossing divisor of multiplicity one is not needed. In place of the composite $\Gamma_i^{\alpha_i}$ there, one could try looking for algebraic cycles in the higher exterior power complexes here rather than only iterating the morphism. Here, where $\mathcal{G}$ has rank one, this is just the fact that $\nabla \circ H$ is multiplication by $n+1$ on $\Lambda^{n+1}\mathcal{P}\mathcal{G}$. It means that H is an embedding of $\Lambda^{n+1}\mathcal{P}\mathcal{G}/torsion$ in $\Lambda^n\mathcal{P}(\mathcal{G}) \otimes \mathcal{G}/torsion$, and in fact the exact sequence going further to the right (with differential H) is exactly like the sequence used in the theory of Castelnuovo-Mumford regularity; it is a torsion-free and exact Koszul complex twisted by tensor powers of $\mathcal{G}$, and it is exact even though $\mathcal{G}$ is an arbitrary torsion free coherent sheaf of rank one. Generically the image is equal to the rank one subsheaf $\Lambda^n(\Omega\mathcal{G}) \otimes \mathcal{G}/torsion = (\Lambda^n\Omega) \otimes \mathcal{G}^{n+1}/torsion$; in fact it is a bit larger. The inclusion composed with the inverse of the isomorphism of H onto its image is precisely the map of theorem 1 for the case $\mathcal{F} = \mathcal{O}_M$.

We see from the observation that if we coordinatize the blowup of $(g_0, ..., g_{n+1})$ we are adjoining an n form with extra poles of degree $n+1$ on the exceptional locus. Without blowing up the ideal, the presence of the additional generators means that we are enlarging the adjunction ideal by an inclusion $I^6 \subset I_6$ without affecting the codimension one primary components (there are none); elements of $\mathcal{F}_6$ are not six-fold symmetric powers of one-forms anymore, they can be viewed as having additional poles on V at primary ideals whose associated prime is the maximal ideal of the cone point.

If $\pi : V' \to V$ is a resolution the sheaf $\oplus_j \pi_*(\omega_{V'}^{\otimes j})$ is just $\oplus_j \omega_V^{\otimes j}$, and any isomorphism $\omega_V \to \mathcal{O}_V$ gives an isomomrphism between this and (the sheafification of) the polynomial algebra $\mathcal{O}_V[T]$. The $\mathcal{F}[N]$ describe a filtration of the graded algebra and we have noticed that $\mathcal{F}_6[4]$ is like six-fold symmetric powers of alternating differential 4-forms with extra poles of codimension four, it is just an integral extension of the un-adorned $\mathcal{F}_6$; while there must exist a number r such that the integral closure of $\mathcal{F}_j[-1]$ is all of $(\mathcal{F}_1^{\otimes j})^{**} = \omega_V^{\otimes j}$ when $r|j$.

10. Divisorial approximation

Let us say that an inclusion $\mathcal{F} \to \mathcal{G}$ of torsion-free coherent sheaves of rank one is *integral* if for each Stein open set U there is some ideal sheaf $\mathcal{Y}$ so that $\Gamma(U, \mathcal{Y}\mathcal{F}) = \Gamma(U, \mathcal{Y}\mathcal{G})$.

The construction of the relative canonical model by taking direct images suggests an analagous construction, and up to integral morphisms this is possible.

Let K_i be the Cartier divisor on U which is the pullback modulo torsion of $\Lambda^n \Omega_{M_i}$. Fix a natural number $N \geq -1$. Let $\tau : U \to M$ be the structural map. Define Cartier divisors $D_1, D_2, ...$ on U to be the integer linear combinations of the K_i given by the equation

$$D_{(N+2)^j} = K_j + (N+1)(K_{j-1} + (N+2)K_{j-2} + (N+2)^2 K_{j-3} + ... + (N+2)^{j-1} K_0)$$

and when $i = a_0 + a_1(N+2) + ... + a_s(N+2)^s$ with $0 \leq a_i < (N+2)$ let

$$D_i = a_0 D_1 + a_1 D_{(N+2)} + ... + a_s D_{(N+2)^s}.$$

If there is any ambiguity about the choice of N we will write $D_i[N]$ to denote the relevant number N which was used in constructing D.

14. Theorem. Let $\mathcal{F} = \mathcal{O}_M$.

a) For each N and j There is a natural map of torsion-free coherent sheaves of rank one

$$\mathcal{F}_j(N) \to \tau_* \mathcal{O}_U(D_j[N]).$$

It is an integral map for all N and j. Therefore $\mathcal{F}[N] \subset \oplus_i \tau_* \mathcal{O}_U(D_i[N])$ is an integral map of $\mathcal{O}_M$-algebra sheaves and, more strongly, of underlying coherent sheaves.

b) For each N and each sequence of numbers increasing with divisibility $i_1|i_2|i_3|...$ the sequence of $\mathbb{Q}-$ divisors on U is an ascending sequence

$$\frac{1}{i_1} D_{i_1}[N] \leq \frac{1}{i_2} D_{i_2}[N] \leq \frac{1}{i_3} D_{i_3}[N], ...$$

and on any bounded subset of $Y_0[N]$ it is a finite ascending series of $\mathbb{Q}$-divisors. Therefore for each N there is a limiting $\mathbb{Q}$-divisor $D[N]$ on U such that at each point of U there is an index r such that $D_i[N] = i \cdot D[N]$ whenever $r|i$.

c) For $N = -1$ the divisor $D[N]$ is merely the canonical divisor K_U of U. That is, $D[-1] = K_U$.

d) Let $\pi : M' \to M$ be any resolution. For each i the pushforward $\tau_* \mathcal{O}_U(iK_U)$ is the same as $\pi_* \mathcal{O}_M(iK_M)$. We can therefore use U in place of M' and $D[-1]$ in place of $K_{M'}$ in defining the homogeneous coordinate ring sheaf of V.

e) Recall $D_i = D_i[n]$. The fibers of $U \to Y_0$ are (complete) projective varieties. Let C be a complete curve in U. Then C is contained in just one fiber of $U \to Y_0$ if and only if the right side of the equation

$$\boxed{K_U \cdot C = -(n+1) \cdot \sum_{i=0}^{\infty} D_{(n+2)^i} \cdot C}$$

has only finitely many nonzero terms. When C is not contained in a fiber the right side is eventually a geometric series. In every case the equation remains valid formally; i.e. as the sum of a convergent series of p-adic numbers for any prime divisor p of $n+2$.

15. Corollary. The 'discrepancy' $K_U \cdot C - K \cdot C$ is the sum over points $p \in C$ of the $gen_p(C) - 1$ where gen_p is the local generating number at p. This differs from the actual discrepancy by $(\tau^* K_M - K) \cdot C$. I think that the sum is the dimension of the vector space $(i^* \mathcal{F}/torsion) \otimes_{\mathcal{F}} \mathcal{O}_M$ if $i : C \to U$ is the map. If γ does not lift, the cardinal dimension of the vector space does not carry full information; the Grothendieck class of the module must be represented by a Poincare series or a p-adic number in that case.

Proof of d). Assume M is contained in a disk. Choose a global section of the i'th power of the canonical sheaf of the resolution M'. Choose a point of U, and choose a stage of the Nash tower $M_m \to M$ which contains a neighbourhood T of this point. The form on the regular locus of M' pulls back to a form on a resolution of the pullback of M' and M_m over M and forward by an isomorphism between an open set and the complement of a codimension two subset of T, and then extends across all of T. For the converse use Hilbert's basis theorem for subsheaves of of $(\Lambda^n \Omega_M^{\otimes i})^{**}$. Replacing M by a bounded open subset if necessary, there is a number m so $\tau_* \mathcal{O}_U(iK_U) = \tau_* \mathcal{O}_U(iK_m)$. In the case M_{m+1} is normal, since $\mathcal{O}_U(iK_m)$ agrees on the regular locus of M_{m+1} with the pullback modulo torsion of $\Lambda^n \Omega_{M_m}^{\otimes i}$, we have a section of a locally free sheaf, which then extends across the codimension-at-least-two singular locus of M_{m+1}. The extended section belongs to the larger sheaf $\Lambda^n \Omega_{M_{m+1}}^{\otimes i}/torsion$ and does pull back to a section of ω^i on a resolution. If M_{m+1} is not normal, boundedness of M_{m+1} in the larger manifold implies that there is an upper bound on the number of initial consecutive finite and nontrivial Nash blowups of any open subset of M_{m+1}. A higher Nash blowup $M_N \to M_{m+1}$ yields, by deleting an exceptional set $E \subset M_N$ and its image $C \subset M_{m+1}$ of codimension at least two, a finite map which is a resolution. Our section restricts on $M_N \setminus E \subset U$ to a section of the pullback of $\Lambda^n \Omega_{M_m}^{\otimes i}$ modulo torsion. Except on the inverse image of C this agrees with a section of the of the (locally free) pullback modulo torsion of $\Lambda^n \Omega_{M_m}^{\otimes i}$ to the normalization of M_{m+1}. It extends to a global section as in the normal case.

It is tempting to summarize the results by saying that $D[n+1]$ is relatively ample on U, $D[n]$ is relatively eventually basepoint free, and $D[-1]$ is canonical with classical relative plurigenera (the same as a resolution). A difficulty is that we have to clarify what 'ample' and 'eventual' actually mean, because both definitions implicitly refer to an index $r(p)$ associated to each point $p \in U$. In the algebraic case semicontinuity of the best $r(p)$ in the Zariski topology and quasicompactness give a bound and it had not been necessary to distinguish the cases. Here we need to clarify whether we intend the $r(p)$ to be bounded above. The assumption that M is a manifold is not important since any singular manifold is contained in a nonsingular manifold.

16. Definition (clarification). Let $\tau : U \to M$ be a map of smooth complex manifolds.

a) A Cartier divisor (or $\mathbb{Q}$-divisor) D on U will be called *uniformly relatively ample* if some multiple rD is relatively very ample.

b) It will be called *non-uniformly relatively ample* if nevertheless for each point $p \in U$ there is an index $r(p)$ depending on p and neighbourhoods T of p and S of $\tau(p)$ such that $\Gamma(S, \tau_* \mathcal{O}_U(rD))$ separates points and tangent vectors on T; ie that the blowup $Bl_{\tau_* \mathcal{O}_U(r(p)\cdot D)} \to M$ contains a copy of T and the restriction of τ to T is induced by the structural map of the blowup.

c) It will be called *uniformly eventually relatively basepoint free* if there is an index r such that $\mathcal{O}_U(rD)$ is relatively basepoint free.

d) It will be called *non-uniformly eventually relatively basepoint free* if nevertheless every point p has in index $r(p)$ depending on p and neighbourhoods T of p and S of $\tau(p)$ such that the restriction map $\Gamma(\tau^{-1}(S), \mathcal{O}_U(rD)) \otimes \mathcal{O}_T \to \mathcal{O}_T(rD))$ is surjective.

The terminology is chosen so that the uniform case is included in the uniform case, characterised by the boundedness of the $r(p)$.

In this sense then

17. Summary. The Cartier $\mathbb{Q}$-divisors $D[N]$ on the manifold U are such that

$$D[N] \text{ is } \begin{cases} \text{non-uniformly relatively ample,} & N \geq n+1 \\ \text{non-uniformly relatively eventually basepoint free,} & N \geq n \\ \text{canonical with classical relative plurigenera,} & N = -1 \end{cases}$$

Things can be made absolute rather than relative in the case M is quasi-projective. We let $\mathcal{F} = \mathcal{O}_V(H)$ for H a hyperplane section. Then, for example

18. Theorem (some global variants) Suppose M is quasi-projective and H is very ample on M. Then

a) Each divisor $D_i + (n+2)^i(n+1)\tau^*H$ is absolutely basepoint free on U.

b) For each $N \geq n+1$ and every point $p \in U$ a suitable integer multiple of the $\mathbb{Q}$-divisor $D[N] + (n+1)\tau^*H$ defines a projective embedding of a neighbourhood of p.

c) For $N = n$ a suitable integer multiple of $D[N] + (n+1)\tau^*H$ depending on p has base locus disjoint from p

d) For $N = -1$ and for all j the global sections on U $\Gamma(\mathcal{O}_U(j(D[-1] + (n+1)\tau^*H))$ are the just the global sections $\Gamma(M', j(K_{M'} + (n+1)\pi^*H)$ for a resolution $\pi : M' \to M$.

e) If M is normal, the fiber in U over each singular point of M always contains at least one contractible complete curve C with $D[n] \cdot C = 0$.

Part d follows from Theorem 14 by the projection formula.

19. Remark. It appears also to be true in the toric case if one makes the strange assumption that all the $\Lambda^n\Omega_{M_i}/torsion$ on the terms of the Nash tower are reflexive then for some reason the basepoint-free results extend all the way down to $D[0]$.

11. Questions

The obvious idea, before trying calculations, is to look for successively stronger basepoint free theorems to attempt to find a series of proper maps in either direction relating Y_0 with V, following Mori's discovery [13]. If M locally admits a smooth one-dimensional foliation, for example, then the next map $Y_0[n] \to Y_0[n-1]$ is also a locally projective holomorphic map. Note that the locally projective morphisms are in the opposite direction from the affine morphisms. In general there is no idea what modification of $Y_0[n]$ can creates a useful relatively-basepoint-free divisor. Theorem 1 is merely tautological, as opposed to more substantial basepoint free theorems in algebraic geometry.

If M is not only a complex manifold but a singularly foliated complex manifold then n may be replaced not by the codimension but the dimension of the foliation, and all the above holds, except it refers not to the Nash manifold, but to the smoothly foliated singular manifold which results by repeatedly blowing up the foliation itself.

20. Conjecture. Suppose M is a complex vector space singularly foliated by the faithful split action of a commutative Lie algebra. Then the foliation is resolvable by a locally projective degree one morphism from a singular variety with a nonsingular foliation if and only if the set of roots (ignoring multiplicity) forms a basis of the dual of the Lie algebra.

This is worked out in the case of toric resolutions in an unpublished arXiv preprint [19]. The same preprint claims without proof

21. Theorem. Suppose M is a complex singular projective variety with very ample divsor H and that M has a singular foliation of dimension s such that the canonical divisor K of the foliation satisfies that $K + (s+1)H$ is a finitely-generated divisor of Iitaka dimension one. Then the foliation can be resolved by one or more Nash blowups of the foliation.

This is true because the successively higher Nash blowups are induced by maps to algebraic curves W_i. The rational function fields $\mathbb{C}(W_i)$ are all contained in a subfield of a field of transcendence degree one. Therefore the rational function fields of the W_i stabilize and then the W_i are bounded by the unique birational model.

Note also that even though the ring $\oplus_i \Gamma(\mathcal{F}_i)$ may conceivably fail to be finite type, the sheaf $\oplus_i \mathcal{F}_i$ actually must then be finite type. That is, since the $Bl_{\mathcal{F}_i} M$ are eventually isomorphic over M for large i, the $\mathcal{F}_i$ pull back to invertible sheaves on suitably high Nash blowups. By Theorem 1 then when i is a large power of $N+2$, $\mathcal{F}_i^{N+2} \to \mathcal{F}_{i(N+2)}$ is onto.

The canonical sheaf is the simplest primary component of the highest exterior power of the differentials. This component is invertible precisely when $-K_0$ is an effective exceptional divisor in the Nash tower. In general, if it is not invertible, if we blow up only this primary component the resulting map is not the identity; if we do so repeatedly

22. Conjecture. If we blow-up only upon the codimension one primary component of $\Lambda^n \Omega / torsion$ at each stage,ie its reflexivication, then after finitely many steps the reflexivication will be invertible; there results a proper morphism from a singular manifold with invertible canonical sheaf.

The conjecture is easily verified to be true for toric surfaces. The pullback modulo torsion of the canonical sheaf is an invertible subsheaf of the reflexive canonical sheaf, and so there is an effective Weil ramification divisor. The process finishes if and only if the Weil ramification becomes zero and that is what the conjecture asserts.

Hironaka's and Spivakovsky's theorem is that for surfaces normalized Nash blowups do yield a proper map of a smooth manifold to M. There appears to be a great distance between the normalized and non-normalized Nash blowups.

When $\mathcal{F}$ is taken to be $\mathcal{O}_M$ then functoriality in the title is with repsect to degree-one holomorphic maps of M. Strictly speaking it applies only to $\mathcal{F}[N]$ for $N \geq n$, after passing to integral closure. For example, the Grauert-Riemannschneider sheaf is the integral closure of $\mathcal{F}_1[-1]$ when $\mathcal{F} = \mathcal{O}_M$, and this is not a functor. The failure of functoriality however is only due to the fact that $\mathcal{H}om$ is contravariant in one of its arguments.

That is, a corollary of part d) of Theorem 14 and the definition of $X_i[-1]$, yields an elementary calculation of Grauert-Riemenschneider's sheaf whenever $\mathcal{F}$ is invertible (e.g. if $\mathcal{F} = \mathcal{O}_M$)

23. Theorem. Suppose M is bounded (within a possibly larger manifold of the same dimension). The Grauert-Riemenschneider sheaf of M is the integral closure of $\mathcal{F}^{-n-1}\mathcal{H}om(\mathcal{F}_{(n+2)^i-1}, \mathcal{F}_{(n+2)^i})$ for all sufficently large value of i.

The calculation of the sheaf has already been within the range of computer since resolution of singularities algorithms exist. A simpler method would be to use the formula above if there were a way of determining what value of i gives the largest answer.

24. Remark. Since for all j there is an i so that the pushforwards to Y_0 of $\mathcal{O}_U(jK_U)$ is dual to the pullback from M to Y_0 of $\mathcal{F}^{\otimes j}_{(n+2)^i-1}$ times the inverse of the pullback modulo torsion of $\mathcal{F}^{\otimes j}_{(n+2)^i}$ then it is a reflexive sheaf for all j on any bounded open subset of Y_0. According to a question of [7] then it is not known whether Y_0 has canonical singularities.

12. Appendix

We will finish this note by describing the proof of properness of $V \to M$. In this section we assume M to be normal and locally algebraic. Refer to [21,22,23] for the full list of references and full statements of theorems. Recall $\pi : M' \to M$ is a resolution.

Kempf describes things this way in his article which happens to be in the Springer Lecture notes [6] about toroidal embeddings: although we have not assumed $0 = R^i\pi_*\Lambda^n\Omega_{M'}$ for $i \geq 1$, Grauert-Riemenshneider vanishing says this is true.

We may assume M is a closed analytic subvariety of a disk A by an embedding i of codimension c. The sheaf $\Lambda^{dim(A)}\Omega_A$ is isomorphic to $\mathcal{O}_A$ but not canonically and we shall be needlessly rigorous about the notation by distinguishing them. Since all higher derived functors vanish, duality simply implies $\mathcal{E}xt^c_{\mathcal{O}_A}(-, \Lambda^{dim(A)}\Omega_A)$ interchange $i_*\pi_*\mathcal{O}_{M'}$ and $i_*\pi_*\Lambda^n\Omega_{M'}$. The isomorphism $\mathcal{O}_M \to \pi_*\mathcal{O}_{M'}$ coming from normality of M gives when we apply i_* and apply our $\mathcal{E}xt$ functor

$$i_*\pi_*\Lambda^n\Omega_{M'} \cong \mathcal{E}xt^c_{\mathcal{O}_A}(i_*\pi_*\mathcal{O}_{M'}, \Lambda^{dim(A)}\Omega_A)$$

$$= \mathcal{E}xt^c(i_*\mathcal{O}_M, \Lambda^{dim(A)}\Omega_A)$$

$$= i_*\mathcal{E}xt^c(\mathcal{O}_M, \Lambda^{dim(A)}\Omega_A)$$

Removing the i_* this is the double dual of the highest exterior power of differentials of M, ie the result of removing all but codimension one primary components. Thus this is the pushforward of the highest exterior power of the differentials of M'

$$\cong i_*\pi_*\Lambda^n\Omega_{M'}.$$

The definition of rational singularities doesn't imply the $\pi_*\mathcal{O}_{M'}(iK_{M'})$ for $i \geq 2$ are the sheaves associated to the divisors iK where K is the canonical divisor. The duality argument above does not extend to the higher values of i. The condition that $\pi_*\mathcal{O}_{M'}(iK_{M'}) = \mathcal{O}_V(iK_M)$, with the extra condition of $\mathbb{Q}$-Cartier (but note it is stated in [7] that the extra condition may be automatic) is the definition of M having canonical singularities.

Always one inclusion holds $\pi_* \mathcal{O}_{M'}(iK_{M'}) \subset \mathcal{O}_V(iK_M)$, and this is natural independent of choice of M'. This is because M is normal, and any Weil divisor is then Cartier except on a locus of codimension at least two. To see this, we can assume we are talking about an irreducible Weil divisor which for a normal variety corresponds to a symboic power of a height one prime ideal. Choosing an element of the local ring at that prime which generates the corresponding power of the prime ideal, this defines the same divisor except on a locus of vanishing of a single element, and the divisor of this element in each affine coordinate chart of a finite cover of the divisor meets the divisor in codimension at least two.

Then the blowup of any sheaf of ideals, by taking a primary decomposition, we see it is an isomorphism away from a codimension two locus. Or we could have chosen our resolution to be an isomorphism away from the codimension at least two singular locus. In any case then the direct image of $\mathcal{O}_{M'}(iK_{M'})$ agrees with $\mathcal{O}_V(iK_M)$ except in codimension at last two. The latter has no primary components with associated primes of height other than one, and so cannot be made larger without affecting something codimension one.

Shepherd Barron in dimension 3 [8] and Elkik in dimension ≥ 4 [9] answered one of the questions in [7] namely that the Cohen Macaulay property, vanishing of all but one $\mathcal{E}xt$, does follow from the conditions of canonical singularities.

Assume the relative minimal model program as described in the conjecture of [11] holds for the resolution of $\pi : M' \to M$. So we are assuming Mori's sequence of contractions (followed by passing to the relative canonical model) [13] leading from M' to a variety $q : M_t \to M$ which has K_{M_t} relatively nef over M and terminal singularities. The relative basepoint free theorem as described in the textbook [21], page 94, even in the analytic case, gives some iK_{M_t} relatively basepoint free. Choose a resolution $h : M'' \to M_t$ of M_t; one has

$$h_* \mathcal{O}(iK_{M''}) = \mathcal{O}(iK_{M_t})$$

By relative basepoint freeness this is

$$= q^* q_* \mathcal{O}_{M_t}(iK_{M_t})$$

$$= q^* q_* h_* \mathcal{O}(iK_{M''})$$

By uniqueness (which uses normality of $M,$ that $iK_{M'}$ is Cartier and contravariant) this is

$$= q^* \pi_* \mathcal{O}(iK_{M'}).$$

$$= q^* \omega_i$$

It follows from these (inefficient) formulas that

$$q_* \mathcal{O}_{M_t}(iK_{M_t}) = \omega_i.$$

The relative basepoint freeness holds with i replaced by $2i, 3i, ...$ because all it is saying is (assuming as we may that M is affine) that at each point of M_t there is a global section with the correct order of pole there, and powers of global sections give the correct order for multiples of the divisor. Therefore there is a continuous function $M_t \to V$ over M and compact subsets of M lift to compact subsets of M_t which remain compact in $V.$

References

1. Michael Atiyah, Complex analytic connections in fibre Bundles, Transactions of the American Mathematical Society, Vol. 85, No. 1, 181-207 (1957)

2. Alexandre Grothendieck, Elements de geometrie algebrique II, publ IHES 8, 5-222 (1961)

3. Heisuke Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero II, Annals of Mathematics 79, 205-326 (1964)

4. John Nash, unpublished (circa 1965)

5. Michel Artin, Algebraic approximation of structures over complete local rings, IHES 36, 23-58 (1969)

6. George Kempf , Finn Knudsen, David Mumford, Bernard Saint-Donat, Toroidal Embeddings 1, Lecture Notes in Mathematics , Vol. 339 (1973)

7. Miles Reid, Canonical 3-folds, proceedings of the Angiers 'Journees de Geometrie Algebrique' (1979)

8. Nick Shepherd-Barron, PhD thesis, Warwick University.

9. Renee Elkik Rationalite des singularites canoniques, Invent. Math. 64 (1981)

10. Heisuke Hironaka, On Nash blowing-up, in Arithmetic and geometry, Vol. II, 103.111, Progr. Math., 36, Birkhauser, Boston, Mass., (1983)

11. Miles Reid, Decomposition of Toric Morphisms, in Arithmetic and Geometry, vol. II: Geometry, Prog. Math. 36,. 395-418 (1983)

12. Helene Esnault, Eckart Viehweg, Logarithmic deRham complexes and vanishing theorems, Invent. Math 86, 191-194 (1986)

13. Shigefumi Mori, Birational classification of algebraic threefolds, International Congress of Mathematicians, Kyoto (1990)

14. Mark Spivakovsky, Sandwiched surface singularities and desingularization of surfaces by normalized Nash transformations, Annals of Mathematics, Vol. 131, No. 3, 411-491 (1993)

15. John Nash, Arc structure of singularities, Duke Math. J. Volume 81, Number 1, 31-38 (1995)

16. Mauro Beltrametti, Andrew Sommese, The adjunction theory of complex projective varieties, DeGruyter (1995)

17. John Moody, On resolving singularities, Journal of the London Mathematical Society, 64 , pp 548-564 (2001)

18. Shihoko Ishii, Janos Kollar, The Nash problem on arc families of singularities Duke Math. J. Volume 120, Number 3,601-620 (2003)

19. John Moody, Finite generation and the Gauss Process, preprint (2004)

20. Janos Kollar, Lectures on resolution of singularities, Annals of Mathematics Studies 166 (2007)

21. Janos Kollar, Shigefumi Mori, Birational geometry of algebraic varieties, Cambridge University Press (United Kingdom), (2008)

22. James McKernan, Recent advances in the MMP after Shokurov, II, (lecture)

23. Caucher Birkar, Paolo Cascini, Christopher Hacon, James McKernan, Existence of minimal models for varieties of log general types, J Amer Math Soc 23 (2010) 405-468