

Riemann Roch following Fulton

Let $M \subset \mathbb{P}^N$ be an n dimensional smooth complex projective variety. For each locally free coherent sheaf \mathcal{F} on M let $t(\mathcal{F})$ be the element of the Chow ring of M corresponding to the formal product of $\alpha/(1 - e^{-\alpha})$ over the Chern roots α . Note if N is locally free with dual sheaf \mathcal{N} then $t(N)ch(\Lambda^* \mathcal{N}) = c_n(N)$ where odd terms in $\Lambda^* \mathcal{N}$ count negatively; but we can't 'solve for $t(N)$ ' working only in the Chow ring of M . We abbreviate $t(\mathcal{T}_M)$ by $t(M)$.

We cannot exponentiate the normal bundle; instead embed \mathbb{P}^N in $\mathbb{P}^N \times \mathbb{C}$ and blow up M . The proper transform of $M \times \mathbb{C}$ meets the exceptional divisor – the normal bundle N plus a projectivized normal bundle at infinity – such that M is the zero section of the normal bundle. Let i be the inclusion of this zero section, \mathcal{N} the normal bundle, and π the projection. For any coherent sheaf \mathcal{F} on M ,

$$i_* \mathcal{F} = \pi^* \mathcal{F} \otimes \Lambda^* \pi^* \mathcal{N}$$

in the Grothendieck group. The fundamental class of the zero section M in the exceptional divisor is $c_n(\pi^* N)$. Writing $x \cdot M = i_* i^* x$ for $x = ch(\pi^* \mathcal{F})t(\pi^* N)^{-1}$ gives

$$ch i_*(\mathcal{F}) = i_*(ch \mathcal{F} \frac{t(M)}{i^* t(\mathbb{P}^N)}).$$

By the projection formula

$$ch i_* \mathcal{F} = t(\mathbb{P}^N)^{-1} i_*(ch(\mathcal{F})t(M))$$

Moving along to a different level $M \times \{p\} \subset M \times \mathbb{C}$, the same rule holds for the inclusion of M in projective space.

For any $j \geq 0$, the dimension of $\Gamma \mathcal{O}_{\mathbb{P}^N}(j)$ is the number of degree j monomials in $N + 1$ variables, this equals

$$t(\mathbb{P}^N) \cdot ch \mathcal{O}(j),$$

meaning the degree of the discrete part of the product in the Chow ring; note $t(\mathbb{P}^N) = t(\mathcal{O}(1))^{N+1}$. Writing $i_* \mathcal{F}$ in the Grothendieck group as an integer linear combination of such positive $\mathcal{O}(j)$ – by using the principal parts sequence – gives

1. Theorem (Hirzebruch Riemann Roch) For sufficiently positive divisors D the dimension of $|D|$ is one less than the degree of the discrete part of $(1 + D + D^2/2! + \dots)t(M)$.