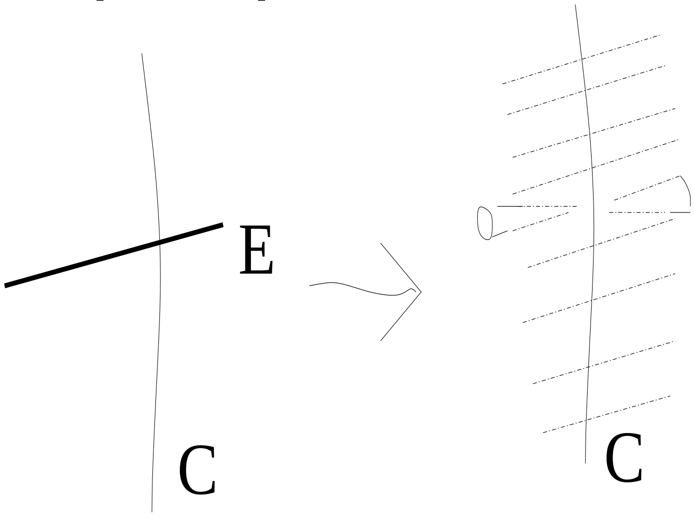
Relations between two singular foliations of a surface. J. Moody

Let V be an affine algebraic surface. Let's look at how the presence of an isolated singularity in V affects the analytic structure of a neighourhood of the singularity. Because this question is not affected by normalization we may assume V is normal.

By resolving the singularity by a map $X \to V$ we analyze the question in five parts. Here is a picture of the main phenomenon where C is an exceptional component and E is a zero of a vector field.



Upon passing from vector field to foliation,

the divisor E disappears leaving only isolated singularities of the foliation. But these determine surrounding geometry and will affect tangency with a second foliation.

1. Let's describe the singularities of an individual foliation $\mathcal{F}$ on the variety X in terms of a divisor E associated to the foliation. Since writing this I have seen it is all well known. The divisor is obtained by choosing a derivation δ of the function field over the constants, and taking the smallest divisor E such that $\delta(\mathcal{O}_V) \subset \mathcal{O}_V(E)$.

   (a) Let C be a smooth complete curve and $C \to X$ a nontrivial map. Then

   $$E{\cdot}C = \begin{cases} \mathcal{F} \cdot C + Y - C^{\,2}, & \text{if } \mathcal{F} \text{ does not preserve } C \\ \mathcal{F} \cdot C + K_C, & \text{if } \mathcal{F} \text{ preserves } C \end{cases}$$

   where $\mathcal{F}{\cdot}C$ measures singularities of $\mathcal{F}$ on the curve C and Y measures tangency of $\mathcal{F}$ on C.

   (b) The negative $-E$ is an effective divisor if and only if $\mathcal{F}$ has a generating vector field which is well-defined everywhere, not only a derivation of the rational function field.

   In case $E \cdot C = 0$ these formulas resemble the familiar formulas for the number of fixed points of a tangential or normal flow. Note the right sides are usually positive for numerical reasons.

   Proof: The intersection number is defined to be the degree of the pullback of E to C. We start with a map $\Omega_X \to \mathcal{O}_X(E)$ with torsion cokernel. Tensoring with $\mathcal{O}_C$ gives exact sequence on C defining the coherent sheaf $\mathcal{F} \cdot C$ on C. $i^*\Omega_X \to \mathcal{O}_C(CE) \to \mathcal{F} \cdot C \to 0$. The left term above is the the middle term in the exact sequence $0 \to \mathcal{O}_C(-C^2) \to i^*\Omega_X \to \Omega_C \to 0$. We get either a nonzero map $\mathcal{O}_C(-C^2) \to \mathcal{O}_C(CE)$ with cokernel an extension of $\mathcal{F} \cdot C$ by Y or else a nonzero map $\Omega_C \to \mathcal{O}_C(CE)$ with cokernel $\mathcal{F} \cdot C$.

2. Let's next describe the singularities of an individual foliation *away from* the singular point. Because we are working locally we can and will ignore isolated singularities of our foliations.

   (a) $\mathcal{F}$ gives rise, after contracting a compact divisor, to a foliation nonsingular everywhere in a neighbourhood of a singular point of some variety V if the support of E is compact and can be contracted.

   (b) The foliation comes from an actual *vector field* not vanishing in a neighbourhood of p if and only if the effectiveness and contractibility of E can be arranged simultaneously.

   (c) In the case when $\mathcal{F}$ preserves the curve C the number $\mathcal{F} \cdot C$ calculates the singularities of the foliation which lie on C, with multiplicities, in the usual sense. In the case when C is not preserved the term $\mathcal{F} \cdot C$ again counts singularities of the foliation on the ambient variety which happen to lie on C (with multiplicities). The extra divisor Y counts tangencies.

3. Next we need to describe the relations *between* a *pair of foliations*, $\mathcal{F}_1, \mathcal{F}_2$.

   (a) Let K be a canonical class of X for any irreducible curve C recall the adjunction formula $K \cdot C = K_C - C^{\,2}$. If derivations of the function field are chosen representing each foliation, then K is not only a divisor class but an actual divisor and $E_1 + E_2 - K$ is an effective divisor. The generating derivations are actual vector fields if and only if the coefficients of E_1, E_2 are ≤ 0. This forces the coefficients of K to be ≤ 0.

   (b) The foliations $\mathcal{F}_1, \mathcal{F}_2$ can be made to correspond to a pair of transverse nonsingular foliations in a neighbourhood of a singular point p if the union $E_1 \cup E_2 \cup K$ is contractible to a point. From earlier, therefore, there is a pair of transverse vector fields in a neighbourhood of the point if and only if in addition $-E_1$ and $-E_2$ can be chosen effective. Then note that $-E_1, -E_2$, and $E_1 + E_2 - K$ are three effective divisors adding up to the effective divisor $-K$.

   (c) For any irreducible curve C

$$(E_1 + E_2 - K) \cdot C =$$

$$\mathcal{F}_1 \cdot C + \mathcal{F}_2 \cdot C + \epsilon_1 Y_1 + \epsilon_2 Y_2 + (1 - \epsilon_1 - \epsilon_2) K_C + (1 - \epsilon_1 - \epsilon_2) C^{\,2}$$

   where

$$\epsilon_i = \begin{cases} 1, & C \text{ not preserved by } \mathcal{F}_i \\ 0, & C \text{ preserved by } \mathcal{F}_i \end{cases}$$

Proof. For (a) note $E_1 + E_2 - K = K - (K - E_1) - (K - E_2)$ and

the terms in parentheses are the divisors associated to the kernels (invertible for projective dimension reasons) of the maps $\Omega_X \to \mathcal{O}_X(E_i)$ for $i = 1, 2$. The divisor compares the second exterior power of Ω_X mod torsion with the tensor product of the two line bundles containing it. For (b) just note that $E_1 \cup E_2 \cup (E_1 + E_2 - K)$ is the same as the given support set. Part (c) just follows by plugging into the formula from 1(a) for $E_i \cdot C$ and using the adjunction formula.

4. **Two Examples** In this section we'll look at two examples. Each example admits a pair of singular foliations and we will find in each case they both preserve a rational curve in common. To construct an example with two transverse foliations everywhere it would be necessary to complete these to projective lines and make all the numerology we have described above consistent so these extra curves can be contracted.

   (a) Example. Blow up a nonsingular point in the (x, y) plane to a rational curve C with self-intersection -1. The vector field $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ in the plane generates our first foliation $\mathcal{F}_1$ on the resolution which has no singular points anywhere and is transverse to C. The divisor E_1 associated to our generator is $-C$ itself as one calculates locally by applying the derivation to the coordinate functions.

   $$E_1 = -C$$

   and the formula for $E_1 \cdot C$ in the non-preserving case 1(a) tells us that

   $$-C^{\ 2} = \mathcal{F} \cdot C + Y - C^{\ 2}.$$

   This confirms that the positive divisors Y and $\mathcal{F} \cdot C$ are both zero.

   Now we take a second vector field in the plane, this time use $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$. Now the new foliation $\mathcal{F}_2$ on the resolution preserves the curve C. It has divisor zero as the vector field lifts to a vector field which is only zero at two points $p_1, p_2 \in C$. Thus $E_2 = 0$ and this time the formula 1(a) tells us

   $$0 = \mathcal{F}_2 \cdot C + K_C$$

   This tells us that the two points of C where $\mathcal{F}_2$ is singular come from the negative of the canonical class of C. Note this is the smallest amount of singular locus that is possible.

   Now we consider how these foliations are related. Strictly speaking now we should start working in a larger projective variety containing our resolution. There we have that

   $$E_1 + E_2 - K$$

is effective. To build K we apply our two vector fields to coordinates in the resolution and take a determinant and this gives $K = -C - L_1 - L_2$ where $L_1 + L_2$ are two lines which are the strict transform of $x = \pm iy$. So our effective divisor $E_1 + E_2 - K$ is just

$$L_1 + L_2.$$

Thus the divisor $L_1 + L_2$ is the divisor of tangency of $\mathcal{F}_1$ and $\mathcal{F}_2$. We could now extend this picture, by viewing the L_i as projective lines in a larger surface and try to use the formulas above to obtain a contractible divisor consisting of two rational curves touching at one point. As we have shown, one of the necessary numerical conditions is that both foliations must simultaneously preserve one of the curves and this does happen here.

(b) Example. This time instead blowing up a point in the plane, let us blow up the cone point in the cone on an ellipic curve. So we are considering the cone given by the equation

$$y^2z = x(x-z)(x-2z).$$

We consider this affine surface and blow up the origin. This time the exceptional curve C has genus 1 and $C^{\ 2} = -3$. For our first foliation we'll take that generated by

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

We get $E_1 = -C$. Our equation tells us

$$-C^{\ 2} = \mathcal{F}_1 \cdot C + Y - C^{\ 2}$$

and there are no singularities anywhere.

For the second vector field we use

$$2yz^2\frac{\partial}{\partial x} + z(3x^2 - 6xz + 2z^2)\frac{\partial}{\partial y}$$

which lifts to the resolution (because of the extra factor of z) . We have $E_2 = -2C - L$ now where L is the strict transform of the line $z = 0$. This is a case when the component is preserved. Thus

$$(-2C - L)C = \mathcal{F}_2 \cdot C + K_C$$

and since the left side has degree five this tells us the foliation has a total of five singular or tangential points somewhere on the elliptic curve. Taking as our canonical divisor $-C$ we find that our effective tangency locus, where the two foliations are tangent to each other, must be some divisor linearly equivalent to

$$E_1 + E_2 - K = -2C - L.$$

This will be five lines meeting C at the five singular points of the second foliation.

Because the first foliation preserves these five lines so must the second, and again we are in a situation where there are lines preserved by both foliations.

5. Remarks about Tangency. Here are some remarks concerning numerical ways of calculating the divisor E_1+E_2-K (without choosing generating vector fields of the foliations).

   (a) The curves C in the support of the effective divisor $E_1 + E_2 - K$ are precisely the curves along which the foliations are tangent to each other. For such a curve $\epsilon_1 = \epsilon_2$ because being tangent to each other they must both either preserve or not preserve C.

   (b) Conversely, if both foliations preserve C then C belongs to the support of $E_1 + E_2 - K$. If both foliations do not preserve C then C may or may not belong. If $\mathcal{F}_1 \cdot C = \mathcal{F}_2 \cdot C = Y_1 = Y_2 = 0$ then choosing generating derivations of the function field, a scalar linear combination of these must preserve C, because the lowest-degree terms of both foliations are represented by an element of the same one-dimensional space of global sections in that case (see next paragraph). Therefore, still in the non-preserving case, if one wants a condition which is actually equivalent to saying C belongs to the support of $E_1 + E_2 - K$ it is this: choose a rational function linear combination of the generating vector fields of the two foliations which preserves C and check if it is identically zero on C.

   (c) The class which determines whether a foliation $\mathcal{F}$ preserves C is an element of $H^0(EC + C^{\,2})$ and is easily calculated once a divisor E of $\mathcal{F}$ is known.