Proofs of the theorems stated in 'Affinization'

Discussion of Theorem 1. This is from the reference [17] though another proof is in [19]. A related fact, which applies for higher principal parts also, is that if M is a torsion-free $R-R$ bimodule with a length n filtration, and if left and right actions agree on the associated graded module, and if I an ideal in the integral domain R, then a Lie commutator calculation shows that the left blowup modulo torsion twisted on the right by I^{n-1}, that is, $X := \bigcup_i \frac{I^i}{f^i} M I^{n-1}$ satisfies $XI = Xf$. It is interesting to compare it then with the two-sided blowup modulo torsion of M twisted by I^{n-1} on the left. It is not the same, but the two bimodules have filtrations in which successive quotients agree, and this extends to coherent sheaves. Thus the right twist by I^{n-1} blown up on the left has the same class in the Grothendieck group of coherent sheaves on the blowup manifold of I as the two-sided blow-up twisted by I^{n-1}.

Proof of Corollary 2. Suppose that on each bounded open subset of M there is a number i such that (1) is surjective after multiplying by $(\mathcal{F}\mathcal{G})^i$. The equation (1) tells us that $\tau^*\Omega_{Bl_{\mathcal{F}}M} \to \Omega_{Bl_{\mathcal{F}\mathcal{G}}M}$ becomes surjective upon taking n'th *exterior powers* and reducing modulo torsion. Now we use

Cramer's rule. The same is true even if we set $n = 1$ in (1), ie if we remove the step of passing to an exterior power. That is, whenever $e_1, ..., e_n$ are local sections of $\tau^*\Omega_{Bl_{\mathcal{F}}M}$ and ω is a local section of $\Omega_{Bl_{\mathcal{F}\mathcal{G}}M}$, within the torsion free tensor product $\Lambda^n\Omega_{Bl_{\mathcal{F}\mathcal{G}}M} \otimes \Omega_{Bl_{\mathcal{F}\mathcal{G}}M}$

$$(e_1 \wedge ... \wedge e_n) \otimes \omega = \sum_{i=1}^{n} (-1)^{n-i} (e_1 \wedge ... \wedge \widehat{e_i} \wedge ... \wedge e_n \wedge \omega) \otimes e_i.$$

where the carat indicates a deleted factor.

On the right side of the equation both tensor factors are in the image of τ^*, and then so must ω be (up to multiplying both sides by a suitable rank one sheaf or equivalently by a sheaf of ideals). Choose any map from a smooth complete curve $C \to Bl_{\mathcal{F}\mathcal{G}}M$ such that the composite with τ sends C to a single point. Pulling back (1) with n replaced by 0, the source of the isomorphism is the sheaf of sections of the trivial vector bundle of rank n. Yet the pullback of $\Omega_{Bl_{\mathcal{F}\mathcal{G}}M}/torsion$ maps onto Ω_C; composing maps of sheaves gives Ω_C generated by n global sections. The genus of C must be at least one. Yet τ is locally projective, if no rational curve is contracted it is a finite map by Stein factorization. By the assumption that $Bl_{\mathcal{F}}M$ is normal, τ is an isomorphism. The converse goes by the reverse argument.

Proof of Corollary 3. Apply Theorem 1 when $\mathcal{F} = \mathcal{O}_M$ and $\mathcal{G} = f_*\mathcal{L}$. Pulling back the target of (1) via f and reducing modulo torsion gives $\Lambda^n\Omega_N/torsion$ twisted by $\mathcal{L}^{n+1}$. Tensoring with the relatively very ample sheaf $\mathcal{L}$ then yields a relatively very ample sheaf in the sense we have defined, that is, the pushforward of $\mathcal{L}^{n+2}\Lambda^{n+1}\Omega_N/torsion$ is an integral extension of $\mathcal{G}\Lambda^{n+1}\mathcal{P}(\mathcal{G})$ whose blowup dominates N.

Proof of Theorem 4. We have inductively

$$\mathcal{F}_{(n+2)^{i+1}} = \Lambda^{n+1}\mathcal{P}(\mathcal{F}\mathcal{F}_1...\mathcal{F}_{(n+2)^i})/torsion.$$

We can assume by the inductive hypothesis that $\mathcal{F}_1...\mathcal{F}_{(n+2)^i}$ is generated by global sections, so we need only show that $\Lambda^{n+1}\mathcal{P}(\mathcal{F}\mathcal{G})/torsion$ is generated by global sections when $\mathcal{F}$ is very ample and $\mathcal{G}$ is generated by global sections. Taking $f_0, ..., f_a$ basic global sections of $\mathcal{F}$ and $g_0, ..., g_b$ of $\mathcal{G}$ the expressions

$$\sum_{i=0}^{n} f_{\alpha_0}g_{\beta_0}d(f_{\alpha_1}g_{\beta_1}) \wedge ... \wedge \widehat{d(f_{\alpha_i}g_{\beta_i})} \wedge .... \wedge d(f_{\alpha_n}g_{\beta_n})$$

where $0 \leq \alpha_0 < ... < \alpha_n \leq a$ and $0 \leq \beta_0 < ... < \beta_n \leq b$ can be viewed as spanning global sections of $\Lambda^{n+1}\mathcal{P}(\mathcal{F}\mathcal{G})/torsion$. The expressions are actually multilinear symmetric of degree $(n+1, n+1)$ in the f_α and g_β; since $\mathcal{F}$ is very ample we interpret the f_α as homogeneous coordinates and the way the expressions are to be interpreted is that M is covered by open sets where some $f_\alpha = 1$. One must finish with a calculation to show that the sheaf generated by these global sections really is the sheaf which we claim it to be.

Proof of Theorem 7. The assumption that $Y_0 \to M$ is proper implies that it is compact, and so it is $Proj \oplus_i \mathcal{F}_i$. The next part follows from the relation between Proj of a sheaf generated by global sections and Proj of the global sections.

Proof of Corollary 8. If the map is a morphism then $Y_0 = M$. By theorem 5 the next-to-last stage of the Nash tower is equal to the last stage. This cannot happen unless the tower is trivial and M is smooth.

Discussion of Theorem 10. Proof as given. Also the inclusion stated above the theorem $\mathcal{F}_j[N+1] \subset \mathcal{F}_j[N]$ can be proven in this way: Comparing the factors of $\mathcal{F}_{(n+2)^i}$ which occur in $X_j[N+1]$ and $X_j[N]$, the fact that the highest term in the base $N+2$ expansion of j is at least as high as the highest term in the base $N+3$ expansion of the same number implies $X_j[N+1]$ is a divisor of a power of $X_j[N]$. It follows that we may treat the factors as though they were invertible. The base $N+3$ and $N+2$ expansions of j describe the unique polynomials $P(T)$ and $Q(T)$ with natural number coefficients such that $P(N+3) = Q(N+2)$ while all coefficients of P are less than $N+3$ and all coefficients of Q are less than $N+2$. We may encode the action of multiplication by $\mathcal{F}_j[N+1]$ and $\mathcal{F}_j[N]$ by related integer polynomials, where the coefficient of T^i describes the effect on the exponent of $\mathcal{F}_{(n+2)^i}$. These polynomials are $P(T) + (N+1-n)\frac{P(T)-P(N+3)}{T-(N+3)}$ and $Q(T) + (N-n)\frac{Q(T)-Q(N+2)}{T-(N+2)}$ respectively. In view of the ring structure of $\oplus\mathcal{F}_i$ we need only verify that all coefficients of the former integer polynomial, as well as the remainder, upon division by $T-(n+2)$, are no larger than the latter. So we are done by

Lemma. Let $1 \leq N \leq n$ be natural numbers. Suppose $P(N+3) = Q(N+2)$ where P, Q are natural number polynomials with all coefficients of P less than $N+3$ and all coefficients of Q less than $N+2$. Then the polynomial

$$Q(T)+(N-n)\frac{Q(T)-Q(N+2)}{T-(N+2)}-P(T)-(N+1-n)\frac{P(T)-P(N+3)}{T-(N+3)}$$

has positive remainder upon division by $T-(n+2)$, and the coefficients of the quotient polynomial are natural numbers.

Example. Take $n = 5$, $N = 4$, $P(T) = 1 + T^2$ and $Q(T) = 2 + 2T + T^2$. We have $P(7) = 50 = Q(6)$ so the hypothesis holds. As $N - n = -1$ and $N + 1 - n = 0$ the two polynomials in question are again $P(T) = T^2 + 1$ and $Q(T) - \frac{Q(T)-Q(6)}{T-6} = T^2 + T - 6$. The difference is $T - 7$, when we divide by $T - n - 2 = T - 7$ we obtain the polynomial 1 which has nonnegative coefficients, and the nonnegative remainder of 0.

Proof of Theorem 11. This theorem merely collates things proven elsewhere in the paper.

a) By 14 a) The integral closure of $\mathcal{F}[-1]$ is $\oplus_i \tau_* \mathcal{O}_U(iD[-1])$ By 14 c) $D[-1] = K_U$ and by 14 d) $\oplus \tau_* \mathcal{O}_U(iK_U)$ is the homogeneous coordinate ring of V.

b) From the definition $\mathcal{F}_i[n] = lim_s \mathcal{H}om(X_i[n]^{n+s}, X_i[n]^{n+s} \otimes \mathcal{F}_i/torsion)$. This is contained in the integral closure of $\mathcal{F}_i$ then, and it follows that there is a finite degree one map $Y_0[N] \to Y_0$.

c) Comparing the exponents of $n + s$ and $N + s$ in the definition we see that for $N \geq n$ $\mathcal{F}_i[N+1]$ is an integral extension of $\mathcal{F}_i \otimes X_i[N+1]/torsion$. Therefore the sequence of sheaves $\mathcal{F}_i[N+1]$ define the best simultaneous local principalization of all $\mathcal{F}_i$ which is U. The further integral extension is then an isomorphism since U is a smooth manifold.

d) This is just because we have described a filtration of a graded sheaf of rings.

e) This is by corollary 6.

Proof of Theorem 12. The proof is included. The fact that an integral extension pulls back modulo torsion to an isomorphism is because we can assume we are talking about ideals; then an equation such as IA=IB for ideals of $\mathbb{C}\{\{T\}\}$ implies $A = B$.

Proof of Theorem 14.

a) The fact that $\tau^* \mathcal{F}_j/torsion \cong \mathcal{O}_U(D_j)$ is how the definition of D_j was chosen in the first place. Also if M is compact, or locally on M, D_j is the pullback of a Cartier divisor on a finite stage of the Nash tower. Therefore $\mathcal{F}_j \subset \tau_* \tau^* \mathcal{F}_j/torsion = \tau_* \mathcal{O}_U(D_j)$. Now, $\tau^* \mathcal{F}_j[N]/torsion \cong (\tau^* X_j[N]/torsion)^{N-n} \otimes (\tau^* \mathcal{F}_j[N]/torsion)$ a tensor product of two invertible sheaves, yielding $\mathcal{O}_U(D_{(n+2)^i} + (N-n)(D_{(n+2)^{i-1}} + (N+2)D_{i-2} + ... + (N+2)^{i-1}D_1)$ and the divisor pictured is $D_j[N]$. Then again pushing forward yields an integral extension.

b) The definition of the $D_i[N]$ gives that when $i|j$ $jD_i[N] \leq iD_j[N]$. On a bounded subset of $Y_0[N]$ we have that $D_{(N+2)^{i+1}} = (N+2)D_{(N+2)^i}$, and so the inclusions given are equalities.

c) This follows from the definition.

d) The proof of d) is given in detail subsequent to the statement of Corollary 15.

e) A suitable neighbourhood of any point $p \in Y_0$ is isomorphic over M to a nieghbourhood of a point in $q \in Bl_{\mathcal{F}_{(n+2)^i}}$ for some i, while the inverse image of that neighbourhood in U is the further blowup of $\mathcal{F}_{(n+2)^j}$ for $j = 1, ..., i-1$. The fiber of $U \to Y_0$ over p is then isomorphic to the fiber of the composite blowup over a $q \in Bl_{\mathcal{F}_{(n+2)^i}} M$ which is a projective variety. We may also do the rest of the problem using the replacmenet blowups. If C maps to a point in Y_0 then since $D[n]$ is a pullback from Y_0 we have $C \cdot D[n] = 0$. Otherwise twisting $\mathcal{F}_i$ to be a sheaf of ideals it pulls back on C to an ample invertible sheaf (because it defines a nontrivial projective morphism). Since $D_{(n+2)^{j+i}} = (n+2)^j D_{(n+2)^i}$ The right side of the given equation for $K_U \cdot C$ is

$$-(n+1) \cdot [\sum_{j=0}^{i-1} D_{(n+2)^j} \cdot C + D_{(n+2)^i} \cdot C \cdot \sum_{j=0}^{\infty} (n+2)^j].$$

The sum, interpreted p-adically, is $-1/(n+1)$ and so we obtain

$$-(n+1) \sum_{j=0}^{i-1} D_{(n+2)^j} \cdot C + D_{(n+2)^i} \cdot C.$$

Since C is compact, for one single suitable choice of i we have

$$D_{(n+2)^i} \cdot C = (K_U + (n+1)(D_0 + ... + D_{(n+2)^{i-1}}) \cdot C$$

and everything cancels except $K_U \cdot C$.