Primer on Classical Mechanics

The notion of a one-form

A one-form on a manifold is a measure of distance. Examples are, on a globe of the earth, degrees to the east, radians to the east, degrees to the north, or radians to the north.

It is not possible to know how far to the east one has travelled, just from knowing the starting and ending positions of the trip. One describes this by saying that the one-form of 'degrees to the east' is not exact. Whereas, degrees to the north is the differential of latitude which is a well-defined function.

The one-forms of degrees to the north, and radians to the north, are scalar multiples of each other. Also, radians to the east and miles to the east are multiples of each other but not by a scalar, rather, by the number of miles from the axis which is a smooth function on the manifold.

Every one-form is a finite sum of such re-scaled exact forms on every part of an open cover, and for smooth real manifolds, with an upper bound on the dimensions of the connected components, this implies that every one-form is a finite sum of exact one-forms times smooth functions, sums of parts like we just considered. Just write the function 1 as a sum of parts which are zero except on one of the open sets of a suitable cover, then the one-form (times one) is a sum of one-forms on the separate open parts.

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The universal one-form

Continuing with the example of a globe, the cotangent bundle is four dimensional and is difficult to visualize, but here is a property which any cotangent bundle has: any smooth arc there (parametrized by a closed interval in the real line) has a well-defined measure, a number (an element of the base field), that is specified without any need of choosing a unit of measurement.

Any cotangent bundle has a natural one-form which can be integrated along any such smooth arc to find its measure.

Moreover, for any compact (oriented) surface with boundary in a cotangent bundle, we may integrate around the boundary to obtain an element of our base field which is then a flux through the surface; and we could obtain the same number then by integrating a univeral two-form over the surface. So any such surface has a well-defined flux or signed area.

However, if we look at an example we will see that this is not mysterious at all. For, a section of the cotangent bundle on an open set is no different than an open subset of the original manifold, which is *furnished with a one-form already.* That is, if we take, say, the lower hemisphere of the earth, and furnish it with the one form which measures radians to the east, then we are looking at a section of the cotangent bundle. The canonical length of any arc there is just the measure of the eastward extent of the arc. Any arc can be moved into a section like this if M has dimension larger than two. As long as M has dimension larger than four, any smooth surface can be moved transversely to the fibers and then it is contained in a section too. For example, the lower hemisphere of the globe furnished with the one-form of radians to the east, which we are looking at already, is itself a section of the cotangent bundle[1] and the universal flux through this surface section is just the total eastward extent of the boundary, which is 2π. [2]

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[1]The south pole itself would need to be sent to a point at infinity in its cotangent space. The relevant one-form, if we parametrize things by complex numbers in the obvious way, is the real part of $-\frac{i}{z}dz$ which, appropriately enough perhaps, has a simple pole at the south pole

[2]If it is divided into regions, all the flux is through whichever region contains the south pole in its interior, if there is one.

The notion of a vector field

A vector field on a subset U – it doesn't necessarily have to be an open subset – of a manifold is a derivation relative to the scalars (which sends scalars to zero in other words) from $\mathcal{O}_M$ to $\mathcal{O}_U$. This gives a unique $\mathcal{O}_M$ linear contracting map on cotangent sections $\Omega_M \to \mathcal{O}_U$ and conversely any $\mathcal{O}_M$ linear map gives a vector field on U. If M is included in a Euclidean space E this gives a map from the pullback of Ω_E to $\mathcal{O}_U$ which we know how to visualize as a 'vector in Euclidean space' in a familiar way.

If U is open and we restrict the derivation to U itself we obtain just a derivation of $\mathcal{O}_U$ and in this way it is possible to compose vector fields. The composite is not a derivation but the Lie action is is. The Lie action of δ acting on a vector field τ is defined

$$\delta\tau(f) = \delta(\tau(f)) - \tau(\delta(f)).$$

To avoid confusion with the composite, it is sometimes denoted $[\delta, \tau]$, it is antisymmetric and satisfies Jacobi's identity. However we can use juxtaposition as long as we aren't going to write the composites of any derivations. If a manifold is in a vector space and points are moving with velocity vector δ then velocity vectors τ are points in the vector-space, and their velocity vectors are the Lie derivatives $\delta\tau$.

We may also take Lie derivatives of one-forms. Let $\delta(df) = d\delta(f)$ and extend by Leibniz rule so $\delta(gdf) = \delta(g)df + gd(\delta(f))$. Define the linear contraction operator i associated to δ by $i(gdf) = gd\delta(f)$. This sends one-forms to functions and satisfies[3]

$$\delta\ = d\ i\ + i\ d.$$

We can also define the contracting action of i on alternating forms ω of any degree so that the same continues to hold.[4] Note then $i\ \delta = \delta\ i$. The velocity vector of a one-form under a flow is just another one-form, and the same for alternating forms of any degree.

[3]This is called Cartan's formula; its relevance to the Lagrangian and Hamiltonian formulations of physics is nicely explained in the Utrech Spring school notes of the late Prof. Duistermaat, which are on his web page.

[4]For instance $\delta(dfdg) = d(\delta(f))dg + d(f)d(\delta\ g)$ is d applied to $i(dfdg) = \delta(f)dg - \delta(g)df$.

Two linear operators

Let M be a manifold (real or singular complex) and let $\pi : N \to M$ be the tangent bundle. Now we can introduce two different $\mathcal{O}_N$-linear operators on alternating differential forms on N.

An operator η

The one-forms on N *relative to* M are naturally[5] isomorphic to the one-forms on N which are sections of the *pullback* from M. Hence there is a natural $\mathcal{O}_N$-linear operator η with

$$Kernel(\eta) = Image(\eta)$$

which sends any local or entire one-form to the relative form viewed as a section of the pullback. Note that an individual section of the pullback does not need to be the pullback of an individual section. Note also that η is nilpotent of order two, $\eta \circ \eta = 0$.

The operator j

The Euler derivation ϵ along the fibers of N is also natural, and a second $\mathcal{O}_N$-linear operator is the corresponding contraction j which operates on alternating differential forms of every degree on N.

[5] If M is singular one needs to work modulo torsion etcetra, we'll ignore such adjustments from now on and assume it is not singular

Two conditions.

Definition. A flow on N with contracting operator i is called[6] *involutive*, if

$$i\ \eta = j$$

Definition. A flow δ is *Lagrangian* with respect to closed a one-form ω, if

$$\delta\ \eta\ \omega = \omega$$

[6]In terms of school physics, as confusing as that is, this means that when we look at a point with an arrow emanating from it, as both the point and arrow change, the arrow actually describes the velocity of the point.

Lagrangian flows

The following proposition follows immediately[7] and formally from the definition.

1. Proposition. Let δ be involutive. Then and δ is Lagrangian for ω if and only if

$$i\ d\ \eta\ \omega = \omega - d\ j\ \omega.$$

When this holds the flow preserves the two-form $d\ \eta\ \omega$ just because $d\ \delta = \delta\ d$ and $\delta\ \eta\ \omega = \omega$ is assumed to be closed.

[7]

$$\begin{aligned} i\ d\ \eta\ \omega &= \delta\ \eta\ \omega - d\ i\ \eta\ \omega \\ &= \delta\ \eta\ \omega - d\ j\ \omega \end{aligned}$$

which equals $\omega - d\ j\ \omega$ if and only if $\omega = \delta\ \eta\ \omega$.

The multipicative group action on N

Let M be any manifold, and $N \to M$ the tangent bundle. In classical mechanics, one wants trajectories in the tangent space of an involutive vector field δ on N not to depend on units of time.

Regardless of what the base field may be, let us denote by $\overline{\mathbb{C}}$ the one-dimensional representation of the nonzero scalars such that

$$\mathbb{C} \otimes \overline{\mathbb{C}}$$

has trivial action, where $\mathbb{C}$ is the one-dimensional representation given the ordinary scalar action. The isomorphism between relative one forms and sections of the pullback can be made equivariant

$$\pi^*\Omega_M \cong \Omega_{N/M} \otimes \overline{\mathbb{C}}.$$

A section of the left side which happens to be the pullback of a local one form on M has trivial scalar action, and the tensor product on the right side cancels the scalar action on the fibers to agree with this.

The exact sequence we've already seen has a corresponding equivariant exact sequence

$$0 \to \Omega_{N/M} \otimes \overline{\mathbb{C}} \to \Omega_N \to \Omega_{N/M} \to 0 \quad (1)$$

which agrees with the sequence we've just now considered, but such that the maps commute not only with the scalar action[8] in the definition of a vector-space but also with the scalar action coming from the vector space structure on the fibers of $N \to M$.

Correspondingly there is a map $\eta : \Omega_N \otimes \overline{\mathbb{C}} \to \Omega_N$ with

$$Kernel(\eta) = Image(\eta) \otimes \overline{\mathbb{C}}.$$

We view the contracting map i_δ of a vector field δ as a map

$$i_\delta : \Omega_N \otimes \overline{\mathbb{C}} \to \mathcal{O}_N \qquad (2)$$

and we will say i_δ is *equivariant* if this map is scalar equivariant.

[8]When the field is finite, even if the action on rational points is trivial or if there are no rational points, there is a well-known way of defining the scalar action on the fibers of N using algebra so that there is an algebraic group action on the various coherent sheaves.

‘Dimensional Analysis’

We won’t require the contracting map i_δ of a vector field δ to be equivariant for the scalar action; but those which are are the interesting ones. There is an old concept in physics called ‘dimensional analysis’ which requires that when you specify a quantity like acceleration, you should write beside it a rational monomial describing units of measurement. So for instance, one writes the ordinary acceleration of gravity as 10 $metres/second^2$. This reminds us that if we change from units of seconds to units of milliseconds acceleration vectors are multiplied by the square of that ratio, which is one million. On a cotangent bundle no numerator of ‘metres’ is needed.

For an involutive vector field δ to have equivariant contracting map is equivalent to saying that acceleration coordinates are given locally by quadratic[9] forms in velocity coordinates with smooth function coefficients.

The formulation of familiar problems such as constant acceleration $a = \lambda$ can be made equivariant by writing $a = \lambda s^2$ for s the rate of progress of time.

We don’t assume that contracting maps of actions are scalar equivariant but they always have a scalar equivariant isotypical component corresponding to an underlying equivariant action.

2. Lemma. If there is any involutive δ there is one such that i_δ is equivariant.

This is true because equivariance is just invariance for the action of scalars λ given $\lambda \cdot i_\delta = \lambda i_\delta \lambda^{-1}$. The scalar action is semisimple and there is a projection to the invariants. The quadratic forms correspond to the fact that the contracting map applied to a differential form like ydx is a quadratic form like yx.

[9]Let’s contrast this with the notion of a symmetric connection: a connection $\nabla : \Omega_M \to \Omega_M \otimes \Omega_M$ is symmetric if closed one forms are sent to symmetric tensors. Here the failure of being a connection, the coboundary plus the the involution, will be the antisymmetrization on all one forms therefore the coboundary itself will be symmetric; it will be the symmetrization.

The notion of a connection

Definition. A *connection* on the tangent bundle $N \to M$ is an equivariant isomorphism of exact sequences between (1) and the split sequence

$$0 \to \pi^*\Omega_M \to \pi^*\Omega_M \oplus \Omega_{N/M} \to \Omega_{N/M} \to 0. \quad (3)$$

It is determined by the map u in the reverse direction of η, and in cases when one does not wish to require a entire connection, the *sheaf of connections* on N is isomorphic to the sheaf of equivariant $\mathcal{O}_M$-linear maps u satisfying

$$\eta\ u\ \eta = \eta.$$

To say u is equivariant is just to say

$$u\ \epsilon = \epsilon\ u.$$

Then the sheaf of connections is represented as the sheaf of endomorphisms of Ω_N that satisfy both equations. We didn't really need to introduce the tensor product with $\overline{\mathbb{C}}$ and we could have used the map η which we defined before, but this second equation would be more difficult to state[10].

If M is a real smooth manifold, choosing a Riemannian structure provides one for N. There is an orthogonal complement of the kernel of η, which is the Levi Civita connection. The involutive condition on δ just means that i_δ has been specified on the left summand, and can be an arbitrary equivariant map $\pi^*\Omega_M \to \mathcal{O}_N \otimes \overline{\mathbb{C}}$ on the right summand. Any such map is the same as map $\Omega_M \to \pi_*\mathcal{O}_N$ by the usual adjunction formalism, that is, a section of the pullback of vector fields on M, and equivariance just means that the image of Ω_M belongs to $\mathcal{O}_M \subset \pi_*\mathcal{O}_N$. So that it is the contracting map of just one vector field, not a linear combination.

[10] It would need to say not that $u\ \epsilon$ and $\epsilon\ u$ are equal, but rather we could say that all the various sheaves are graded according to natural numbers which are eigenvalues of the Euler derivation, that whereas η decreases degrees by one, u has to increase degrees by one.

The connection coming from acceleration

If one wishes to use classical physics formalisms on an analytic manifold, one would like an entire involutive vector field δ. (And such a vector field is Lagrangian for an exact $\omega = dL$ if and only if the integral curves are critical for the time integral of L, under deformations of the curve). But we have to consider the question of existence of an involutive vector field δ.

When there is a connection so $u = \eta\, u\, \eta$ we've just seen that $i_\delta = j\, u$ makes δ involutive;[11] the next theorem provides the converse of this argument. It imposes a very serious restriction on which complex manifolds can have an action, but no immediate restriction on real analytic manifolds, they have at least one Stein complexification.

3. Theorem. The tangent bundle N of M has an entire involutive vector field if and only if it has a connection.

We'll prove this in two steps. By Proposition 2.1.1 of Kapranov *Rozansky Witten Invariants* [1] or Proposition 1 of N. Markarian *The Atiyah Class* [2] the Atiyah belongs to the $+1$ eigenspace for the involution of interchanging the two tensor factors. Since the intersection of the $+1$ and -1 eigenspace is zero, we need to show

4. Lemma. The tangent bundle N of M has an entire involutive flow if and only if the Atiyah class $\alpha \in H^1(M, \mathcal{H}om_{\mathcal{O}_M}(\Omega_M, \Omega_M \otimes_{\mathcal{O}_M} \Omega_M))$ belongs to the -1 eigenspace for the involution which interchanges the two factors.

We will give two different proofs of this fact to show various ways of understanding why it is true. The Atiyah class is just the class of the exact sequence we've already looked at, which detects whether the nilpotent map η is semisimple nilpotent.

Because of Lemma 2, the proofs need only consider the case when i_δ is equivariant.

Important notation. From now on when I write $\mathcal{O}_N, \Omega_N, \pi^*\Omega_M$, I am going to mean the sheaf of abelian groups spanned by the quasiinvariant sections in whatever category one is working (continuous, smooth, real analytic, or holomorphic). In other words, the restriction to a radial line in a fiber will be in the algebraic category.

[11] More explicitly, j is zero on the kernel of η, therefore $i_\delta\ \eta = j\ u\ \eta = j\ u\ \eta + j\ (1 - u\ \eta) = j$

First proof. The first proof will use formalities of homological algebra. The extension class of $\eta \otimes \overline{\mathbb{C}}$ belongs to

$$Ext^1_N(\pi^*\Omega_M, \pi^*\Omega_M \otimes \overline{\mathbb{C}}).$$

The Leray spectral sequence of $N \to M$ is trivial since the fibers are Euclidean spaces which have trivial cohomology, so the space above is

$$Ext^1_M(\Omega_M, \pi_*\pi^*\Omega_M \otimes \overline{\mathbb{C}})$$

Equivariance means that it is in the summand

$$\begin{aligned} &Ext^1_M(\Omega_M, \Omega_M \otimes \mathcal{O}_M d(\mathcal{O}_M)) \\ &\quad = Ext^1_M(\Omega_M, \Omega_M \otimes_{\mathcal{O}_M} \Omega_M) \qquad (*) \end{aligned}$$

where the d in the right factor is the deRham differential on M. Now we have the Hochschild[12] spectral sequence

$$H^p(M, \mathcal{E}xt^q_{M\times M}(-\ ,-)) \Rightarrow Ext^{p+q}_{M\times M}(-\ ,-)$$

applied to $(\Omega_M, \Omega_M \otimes_{\mathcal{O}_M} \Omega_M)$ and (*) is the $E_2^{1,0}$ term which maps inectively to [13]

$$Ext^1_{M\times M}(\Omega_M, \Omega_M \otimes \Omega_M)). \qquad (**)$$

which is the first Hochschild cohomology of $\mathcal{H}om_{\mathcal{O}_M}(\Omega_M, \Omega_M \otimes_{\mathcal{O}_M} \Omega_M)$. Here left and right actions are equal so the relevant extension is a 'bimodule' extension, but one made up of two coherent sheaves where the left and right actions agree.

Returning to consider the exact sequence

$$0 \to \pi_*\mathcal{O}_N d\mathcal{O}_M \to \pi_*\Omega_N \to \pi_*\mathcal{O}_N d\Omega_M \to 0,$$

for ω a *local* section of the right side and r a *local* section of Ω_M we write $rd\omega = d(r\omega) - \omega \otimes dr$ with the second error term belonging to the left term of the exact sequence. If we write $c(r)(\omega) = \omega \otimes dr$ then c is an *entire* Hochschild one-cocycle representing an element of (**). From yet a third spectral sequence $H^p(M \times M, \mathcal{E}xt^q(\Omega, \Omega \otimes \Omega)) \Rightarrow Ext^{p+q}(\Omega, \Omega \otimes \Omega))$, a global cocycle represents an E_2 element for the *conjugate* filtration, thus c represents an element of $E_2^{1,0} \cap\ {}_1E_2^{1,0}$.

[12] A. Grothendieck, Sur quelques points d'algbbre homologique, Tohoku Math J. 9 (1957) 119-221

[13] A bit later on I've added a further explanation

As a verification, let's calculate the Hochschild coboundary of c to verify it is zero. Let r_1, r_2 be local sections of $\mathcal{O}_M$ and ω a local section of Ω_M. Then

$$r_1 c(r_2)(\omega) - c(r_1 r_2)(\omega) + c(r_1)(r_2\omega)$$

$$= r_1\omega \otimes dr_2 - \omega \otimes d(r_1 r_2) + r_2\omega \otimes dr_1.$$

This is indeed zero because of Leibniz rule and because the tensor product is over $\mathcal{O}_M$.

Thus we can view $\omega \otimes dr$ as the cocycle representing the Atiyah class. The class is antisymmetric if and only if the symmetrization $\frac{1}{2}(\omega \otimes dr + dr \otimes \omega)$ plus the Hochschild coboundary of a k linear function $\nabla : \Omega_M \to \Omega_M \otimes_{\mathcal{O}_M} \Omega_M$ is zero. The coboundary of ∇ applied to r sends ω to $r\nabla(\omega) - \nabla(r\omega)$. Putting the equations together then, when the coboundary of ∇ is minus the symmetrization, we see that while that k-linear function ∇ is not quite a connection, it satisfies

$$\nabla(r\omega) = r\nabla\omega + \frac{1}{2}(\omega \otimes dr + dr \otimes \omega). \quad (4)$$

Next, we go into the definition of multiplication of polynomials. This is a very trivial fact, but for example on the x, y plane the product of the x and y coordinate functions, evaluated at a point p, is the specialization to the diagonal points (p, p) of the symmetric product xy defined by the rule $xy(p, q) = \frac{1}{2}(x(p)y(q) + x(q)y(p))$. Here we are in a situation where we have just such a symmetric product.

Interpret the *both* local sections dr and ω on the right side of the equation (4) as sections of $\mathcal{O}_N$, and the average as the symmetric product of two one-forms interpreted as a product of sections of $\mathcal{O}_N$. The equation shows that the k-linear function which agrees with d on $\mathcal{O}_M$ and agrees with ∇ on Ω_M satisfies Leibniz rule for the generators of $\mathcal{O}_N$ which belong to the lowest degree components $\mathcal{O}_M$ and Ω_M. The unique extension to a derivation on $\mathcal{O}_N$ is the corresponding involutive vector field δ.

Second Proof.

It is a possible source of confusion, that that even a closed one-form on M is particular type of coordinate function on N, and therefore we may apply the deRham differential d on N without getting zero. Clearing this up a bit actually helps explain very explicitly the geometric meaning of the cocycle defining the Atiyah class. It is a matter of looking directly at Ω_N when $N \to M$ is the tangent bundle of M, or, what is a bit easier, recalling our notational convention that $\mathcal{O}_N$ is spanned by quasi-invariants,

$$\pi_* \mathcal{O}_N = \mathcal{O}_M \oplus \Omega_M \oplus S^2 \Omega_M \oplus \ldots.$$

This is generated by the first two terms as a sheaf of algebras, and therefore

$$\pi_* \Omega_N = (\mathcal{O}_M \oplus \Omega_M \oplus S^2 \Omega_M \oplus \ldots) d\mathcal{O}_M + (\mathcal{O}_M \oplus \Omega_M \oplus S^2 \Omega_M \oplus \ldots) d(\Omega_M).$$

In the first factor, we may replace $d(\mathcal{O}_M)$ by its $\mathcal{O}_M$-linear span, $\mathcal{O}_M d(\mathcal{O}_M) = \Omega_M$ and the product indicated actually is a tensor product then.

However, the second factor is more subtle. What we see is that each product $(S^i \Omega_M) d\Omega_M$ behaves like a tensor product but with an error term. That is, if r is a section of $\mathcal{O}_M$ we have for appropriate local sections a, b

$$ad(rb) = ardb + abdr.$$

The second term belongs to the first factor of our sum decomposition of Ω_N. This explains why Ω_N modulo the first term is a tensor product, that is, the pullback of Ω_M from M.

To lift the tensor product structure to an appropriate subsheaf, which will then be a complement of the kernel, if we wish to do this equivariantly we are required to send the first term Ω_M into the sum

$$\Omega_M \otimes \Omega_M + \mathcal{O}_M d(\Omega_M).$$

This is precisely the principal parts sheaf.

There are now two different ways[14] we are using the same letter d. A section of Ω_M is a linear combination of dx for x a section of $\mathcal{O}_M$, and here the letter d denotes the deRham differential on M. When we apply d again in the right term, we are not taking the deRham differential of a one-form. Rather, we are interpreting each one-form dx as a function on N, and so linear combinations of dx are again functions on N, and then applying the deRham differential on N.

If we remove the coeficient sheaf $\mathcal{O}_M$ in the right term, the decomposition will be a direct sum, but the second term will no longer be an $\mathcal{O}_M$-linear subsheaf. To have an $\mathcal{O}_M$-linear direct sum decomposition we have to replace each $d(\omega)$ by some $d(\omega) - \nabla(\omega)$ such that $\nabla(\omega) \in \Omega \otimes \Omega$ and $d - \nabla$ is $\mathcal{O}_M$ linear. That is,

$$d(r\omega) - \nabla(r\omega) = rd(\omega) - r\nabla\ \omega.$$

This requires the usual rule for covariant differentiation

$$\nabla(r\omega) = r\nabla(\omega) + \omega \otimes dr.$$

Here ω in the second term is a function which is a coefficient of a one-form in the usual way. Note also that when this holds, we have lifted the tensor product structure. Since

$$(d - \nabla)r\omega = r(d - \nabla)\omega$$

the sheaf

$$S^{\cdot}(\Omega_M)(d - \nabla)(\Omega_M) \subset (S^{\cdot}\Omega)d\mathcal{O}_M + S^{\cdot}\Omega_M d(\Omega_M)$$

is a complement to the subsheaf $S^{\cdot}\Omega_M d(\mathcal{O}_M)$, and the reason it is isomorphic to the tensor product is that the condition for ∇ to be a connection contrives that $(d - \nabla)$, being $\mathcal{O}_M$ linear, acts exactly like a tensor product sign.

[14] It is tempting to call them d and d' as one does with the holomorphic and antiholomorphic differentials

If we only wish to produce an involutive vector field with equivariant contracting map, we can push out the exact sequence we have been considering

$$0 \to \pi^*\Omega_M \otimes \overline{\mathbb{C}} \to \Omega_N \otimes \overline{\mathbb{C}} \to \pi^*\Omega_M \to 0$$

We push out along the map

$$j \otimes \overline{\mathbb{C}} : \pi^*\Omega_M \otimes \overline{\mathbb{C}} \to \mathcal{O}_N \otimes \overline{\mathbb{C}}$$

This map is induced by $\Omega_M \to \pi_*\mathcal{O}_N$ the inclusion as the degree-one term, and the image is then the ideal sheaf generated by the degree one part, which is the defining ideal of the zero section $M \subset N$.

If we restrict attention to scalar invariant summand for the scalar action on N we have that in the principal parts sequence

$$0 \to \Omega_M \otimes_{\mathcal{O}_M} \Omega_M \to \mathcal{P}(\Omega_M) \to \Omega_M \to 0$$

the image of the leftmost term under the map induced by j is the second symmetric power $S^2\Omega_M$. The antisymmetric part of the kernel of the principal parts sequence is sent to zero. The pushed-out sequence splits if and only if the extension class is negated upon interchanging the two tensor factors[15].

This completes the proof of the lemma and therefore the theorem. By naturalityh of Chern classes, it implies

5. Corollary. In a compact Kahler manifold with an action, any open subset U must be such that any cycle representing a Chern class is homologous to zero *within* U itself.

For the case of the Euler class, if it is not zero, and an action exists at all but one point, then the vector field at that point isn't allowed to be zero (corresponding to a singular foliation). If $dim(M) > 1$ it must actually be indeterminate.

[15] An amusing way of combining ideas is to consider that both η and j are differentials in locally free exact chain complexes on N

$$\begin{array}{ccccccccc} & \stackrel{\eta}{\to} & \Omega_N & \stackrel{\eta}{\to} & \Omega_N & \to & \Omega_{N/M} & \to & 0 \\ & & \downarrow & & \vdots & & & & \\ \Lambda^2\Omega_{N/M} & \to & \Omega_{M/N} & \to & \mathcal{O}_N & \to & \mathcal{O}_M & \to & 0 \end{array}.$$

Here j is diagonal $\Omega_N \to \mathcal{O}_N$ and induces the lower row which is exact. The dotted arrow would be i, resulting in a map of locally free resolutions of the resulting cokernel map $\Omega_{N/M} \to \mathcal{O}_M$, which is a field of vectors relative to M defined along M.

Chern's character

Let $\mathcal{F}$ be a locally free coherent sheaf on M. As we saw earlier in the case $\mathcal{F} = \Omega_M$, the Leray spectral sequence of $M \times M \to M \to point$ maps $E_2^{1,0} = Ext^1(\mathcal{F}, \mathcal{F} \otimes \Omega_M)$ injectively into $Ext^1_{M\times M}(\mathcal{F}, \mathcal{F} \otimes \Omega_M)$. Atiyah's class mapped to the element represented by the Hochschild cocycle $c(r)(f) = f \otimes dr$. If $\mathcal{F}$ is locally free this is an element of the Hochschild cohomology of $\mathcal{E}nd\mathcal{F} \otimes \Lambda^{\cdot}\Omega_M$ and each shuffle power c^p represents an element of $Ext^p_{M\times M}(\mathcal{O}_M, \mathcal{E}nd\ \mathcal{F} \otimes \Lambda^p \Omega_M)$.

Because it is represented by a global cocycle it is also in the term ${}_1E^{p,0}$ of the *conjugate filtration* for the spectral sequence $H^i(M \times M, \mathcal{E}xt^j_{M\times M}(\mathcal{F}, \mathcal{F} \otimes \Lambda^p \Omega_M) \Rightarrow Ext^{i+j}_{M\times M}(\mathcal{F}, \mathcal{F} \otimes \Lambda^p \Omega_M)$.

And because it is a power of c, it is contained in the term $E_2^{p,0} = H^p(M, \mathcal{E}nd(\mathcal{F}) \otimes \Lambda^p \Omega)$ for the Leray spectral sequence, which by the trace map $End\ \mathcal{F} \to \mathcal{O}_{\mathcal{M}}$ to $H^p(M, \Lambda^p(\Omega_M) = H^{p,p}(M, \mathbb{C})$, gives a (p,p) form if M is compact Kahler, for instance. The shuffle power c^p representing the class in the intersection $E_2^{p,0} \cap \ {}_1E_2^{p,0}$ is just the Hochschild p cocycle of $\mathcal{H}om(\Omega, \Omega \otimes \Lambda^p \Omega)$ sending local sections $r_1, ..., r_n$ to the function $c^p(r_1, ..., r_n)$ defined by the rule that for any local section ω

$$c^p(r_1, ..., r_n)(\omega) = p! \omega \otimes dr_1 \wedge ... \wedge dr_n,$$

and so the cohomology class $\alpha^p/p!$ which is the degree p part of the Chern character is represented by the cocycle sending local sections $r_1, ..., r_n$ to the function sending f to

$$f \otimes dr_1 \wedge ... \wedge dr_n.$$

on $\mathcal{F} \otimes \Lambda^p \Omega_M$. [16]

[16] It may be tempting to formalize this as the 'trace' of the identity endomorphism of $\mathcal{F} \otimes \Lambda^p \Omega_M$, and this can be done because the map c is represented by an alternating form inducing the identity endomoprhism there, and the 'trace' of any other endomorphism ϕ does give an analogous global Hochschild cocycle. However, the elements of $H^{p,p}(M, \mathbb{C})$ that arise can be obtained more easily by just applying the vector space endomorphism $H^{p,p}(M, \mathbb{C}) = H^p(M, \Lambda^p \Omega_M) \xrightarrow{\phi} H^p(M, \Lambda^p \Omega_M) = H^{p,p}(M, \mathbb{C})$ to the ordinary Chern character of $\mathcal{F}$.

Remark. The identity endomorphism of $\Omega_M \otimes \Lambda^p\Omega_M$ sending $\omega \otimes \phi \mapsto \omega \otimes \phi$ if we locally write $\phi = dr_1 \wedge ... \wedge dr_p$ is the Hochshcild p cocycle which is $\frac{1}{p!}$ times the p'th power $c^p(r_1, ..., r_p)(\omega) = d\omega \wedge dr_1 \wedge ... \wedge dr_p$. Writing $\overline{c}(r)\omega = dr \otimes \omega$ then the shuffle power

$$\begin{aligned}\overline{c}^p(r_1, ..., r_p)(\omega) &= \sum_\sigma (-1)^{sign(\sigma)} dr_{\sigma(1)} \otimes dr_{\sigma(2)} \otimes ... \otimes dr_{\sigma(p)} \otimes \omega \\ &= (-1)^{p-1}(p-1)!\ \omega \wedge j(dr_1 \wedge ... \wedge dr_p)\end{aligned}.$$

The occurrences of d to the right of the first tensor sign are the deRham differential on N, we might we use d' in place of d for these so that the Euler contraction j has $j(d'r) = dr$; it gives an alternating sum where one factor is brought to the left to play the role of $dr_{\sigma(1)}$. The conjugate element $(-\overline{\alpha})^p/p! \in H^{p,p}(M, \mathbb{C})$ is the trace of the endomorphism $\omega \otimes \phi \mapsto -\frac{1}{p}\omega \wedge j\phi$. From the rule $(c + \overline{c})\overline{c} = 0$ one has $e^{\alpha+\overline{\alpha}} = 1 + e^\alpha - e^{-\overline{\alpha}}$ and antisymmetry gives $e^\alpha = e^{-\overline{\alpha}}$. Thus antisymmetry of the Atiyah class gives that the Chern character plus the trace of $\frac{1}{p}\omega \wedge j\phi$ is zero, but in fact both terms are zero separately.

Remark. Here is what happens if one starts with a symplectic manifold M such as a cotangent bundle. Then in the exact sequence $0 \to \pi^*\Omega_M \to \Omega_N \to \Omega_{N/M} \to 0$ the end terms are dual to each other as locally free coherent sheaves on N, and so the map $j : \pi^*\Omega_M \to \mathcal{O}_N$ induces a natural global section of $\Omega_{N/M}$. Also there is a symplectic structure on Ω_N coming from the fact that the two end terms are dual. If it is possible to lift the global section of Ω_N/M to a global section of Ω_N then pairing against this extends j to a map i_δ on the whole of Ω_N providing an involutive vector field δ. The cohomology class which obstructs lifting this one vector field is in $H^1(N, \pi^*\Omega_M) = H^1(M, \pi_*\pi^*\Omega_M)$, and if one assumes things are scalar equivariant as we may, it is in $H^1(M, S^2(\Omega_M) \otimes \Omega_M) = H^1(M, \mathcal{H}om(\Omega_M, S^2(\Omega_M))$. It is likely just the Atiyah class again, encountered intact. An abstract isomorphism between Ω_M and its dual need not be sufficiently natural to provide any entire action; and this is so even when it is from a structure of M as a cotangent bundle of, say, M_0, unless the individual Chern roots α of M_0 itself satisfy that the $e^\alpha + e^{-\alpha}$ together add to zero[17].

[17] Let's state this here since we didn't get any chance to use it! $\mathcal{T}or_{M\times M}(\mathcal{O}_M, \mathcal{O}_M)$ is the same as $\Lambda^{\cdot}\Omega_M$.

Real manifolds

Here then is the situation with real analytic manifolds. If M is a real analytic manifold, then there is always an entire action on the tangent bundle. This extends to a holomorphic action on a neighbourhood of the real points in any complexivication. However, applying Coro 5 to a tubular neighbourhood of the real points forces

6. Corollary. If a real action has any smooth invariant[18] manifold with a nontrivial Pontryagin class then the action does not extend to a holomorphic action on (the tangent bundle of) any Kahler compactification of a complexification of M.

By Novikov's theorem, this cannot be corrected by altering the smooth structure of M, and it means that certain topological configurations just never occur in this way.

By Hirzebruch's signature theorem,

7. Corollary. Let M be a real manifold with an action. If any smooth invariant submanifold has nonzero signature, no complexification of the action extends to a Kahler compactification.

If the points of M are configurations of some planets under gravity, the action is only defined when no planet intrudes into the radius of another. Ignoring rotations of the planets the action extends analytically to the complement of the multidiagonal by pretending the masses were concentrated at the centers. It is indeterminate on the multidiagonal, though one may wish to compactify at infinity by a projective space. One might also wish to replace the multidiagonal by blowing up a nonreduced scheme structure in an attempt to remove indeterminacy. This is a Kahler compactification. If any smooth limit cycle has a nontrivial pontryagin class it would then be impossible to extend the action holomorphically without some remaining indeterminate points.

[18]The notion of invariance needs to be clarified when the submanifold has dimension larger than one and is smaller than M. A limit circle in N which maps to its image in M via a covering map has the property that the image in M has tangent bundle an invariant submanifold of N because the action spans the tangent bundle of the submanifold. For higher dimensional invariant submanifolds the proofs require invariance to mean a smooth submanifold of M with tangent bundle a submanifold of N invariant under the flow.

Cycles relative to the boundary representing Pontryagin classes are sets of ordinary configurations of the planets, not anything mysterious having to do in any way with very large or small distances or compactifications, and the action as it is given has no indeterminacy there. Yet a nontrivial Pontryagin class of an invariant manifold implies that no scheme structure on the multidiagonal can be blown up to remove the indeterminacy. So they relate to unresolvability of things which which in real life may be fictitious, infinitesimal or infinitely far away.

It follows that if an invariant manifold, such as a smooth limit cycle, for the n body problem has nontrivial Pontryagin class, and if the action were holomorpic on the divisor at infinity, then there is no scheme structure on the multidiagonal which is invariant for the action and whose blowup (of M) is free of singularities.

This is true because an invariant scheme structure on the multidiagonal would result in a vector field free of poles on the exceptional divisor which is ruled out by the nontrivial Pontryagin class. Note that if the resolution were smooth, the resolution map would just be a map of smooth manifolds whose critical image is contained in the multidiagonal.

But this is a triviality as it is, we should consider analagous statements when the action is allowed to be meromorphic at infinity.

More about the Lagrangian Condition

8. Theorem If δ is an involutive vector field on the tangent bundle $\pi : N \to M$, and i_δ is scalar equivariant, then the one forms with respect to which δ is Lagrangian are precisely the closed forms in $\delta(\pi^*\Omega_M)$.

Proof. First let's show that $\eta\ \delta$ acts by the identity on $\pi^*\Omega_M \subset \Omega_N$. If m is a monomial in local sections of $\mathcal{O}_N$ and f a local section of $\mathcal{O}_M$, using d' for the differential on N, then

$$\begin{aligned} \eta\ \delta(md'f) &= \eta\ (\ \delta\ (m)d'f + md'\delta f\ ) \\ &= \eta\ (\delta(m)d'f + md'df\ ) \end{aligned} .$$

The first term is zero since η is $\mathcal{O}_M$ linear and $\eta\ d'f = 0$. In the second term, $\eta\ (d'df) = d'f$. Such products $md'f$ span $\pi^*\Omega_M$ as a sheaf of abelian groups.

Now if τ is a local section of $\pi^*\Omega$ and $\delta\ \tau$ is closed, it is fixed by $\delta\ \eta$ and so δ is Lagrangian for $\omega = \delta\ \tau$. Conversely, for an arbitrary closed ω with $\omega = \delta\ \eta\ \omega$ let $\tau = \eta\ \omega$ and one sees that $\omega = \delta\ \tau$.

What this means is that for a closed form ω, as long as δ is involutive and i_δ is scalar equivariant, the condition for δ to be Lagrangian for ω can be simplified, it is enough to check that ω is in the image of $\delta\ \eta$.

Hamilton's equation

Choose a closed one form ω on the tangent bundle N of M. Locally write

$$\begin{cases} \beta = \eta\ \omega \\ dH = \omega - d\ j\ \omega \end{cases}$$

so that the equation in proposition 1. becomes

$$i\ d\ \beta = dH.$$

The same equation can be formulated not only when N is a tangent bundle, but also on any manifold N and for any one-form β whatsoever. As a local equation it is equivalent to $d\ \beta$ being preserved under the flow of the vector field δ whose contraction operator is i. So that the flux through a surface is the same as through any slice of the local orbits. If we were to apply i again since $i\ i = 0$ we obtain $0 = \delta(H)$, showing that the value of H is conserved under δ. however, the equation says more without applying i again.[19] [20]

[19]When N is a cotangent bundle and β the universal one-form, we can see this explicitly. If M has coordinates x_i write points of the cotangent bundle as $(q_1, ..., q_n, \sum p_i dx_i)$. Any smooth function $H(q_1, ..., q_n, p_1, .., p_n)$ has differential

$$\sum (\partial H/\partial q_i) dq_i + (\partial H/\partial p_i) dp_i.$$

Taking for β the canonical form $\sum p_i dq_i$ the contraction of the differential $i\ d\ \beta$ is

$$\sum_i \delta(p_i) dq_i - \delta(q_i) dp_i.$$

In this setting, our equation identifying dH with $i\ d\ \beta$ gives the Hamilton equations

$$\begin{cases} \delta(p_i) = \partial H/\partial q_i \\ \delta(q_i) = -\partial H/\partial p_i \end{cases}$$

solving the ordinary system of differential equations which underlies δ.

[20]Note in the previous footnote that p_i is really just $\partial/\partial q_i$, an ordinary coordinate function on the cotangent bundle, and the canonical form is $\sum \partial/\partial q_i dq_i$. This notation rapidly becomes confusing, such as writing $d\partial/\partial q_i$ for a one-form on the cotangent bundle.

The Hamiltonian formulation – what it is meant to do

This section is taken from a later paper, Standing Waves, but it is of relevance here too.

Firstly, our notation can be simplified if we allow throughout both the deRham differential d of M and the one d' of its tangent bundle N. Recall that ϵ is the Euler derivation, j its contraction operator, and that a vector field δ is *involutive* if $i_\delta \eta = j$, and it is *Lagrangian* for a closed one-form ω on N if $\delta\eta\omega = \omega$.

It is easy to see that the involutive condition just means that when f is a section of $\mathcal{O}_M$ viewed as a function on N constant along fibers, then $\delta(f) = df$, where we view df as a section of $\mathcal{O}_N$.

If one examines various statements that Lagrangian forms make certain integrals critical, they just come down to conditions which are satisfied just for Lie derivatives of one-forms. They are nothing but criteria for recognizing when a one-form happens to be a Lie derivative with respect to a particular vector field, and they do not need to be considered if one is already happy with the concept that one forms have Lie derivatives.

In coordinates, locally, a section of Ω_N can be written

$$\omega = \sum_i p_i d' dq_i + r_i d' q_i$$

with p_i, r_i sections of $\mathcal{O}_N$ but q_i are only sections of $\mathcal{O}_M$. Then $\eta\omega$ is only

$$\sum_i p_i d' q_i$$

an arbitrary section of the pullback of one forms from M. And as long as δ is involutive, when we write

$$\delta\eta\omega = \sum_i \delta(p_i) d' q_i + p_i d' \delta(q_i)$$

we can rewrite the last term

$$= \sum_i \delta(p_i) d' q_i + p_i d' dq_i.$$

To say that this equals ω again is to say

$$r_i = \delta(p_i).$$

This is just the condition for ω to be the Lie derivative of a one-form pulled back from M. But if ω is exact locally so it is dL for a function L on M, then

$$p_i = \frac{\partial}{\partial(dq_i)} L$$

$$\delta(p_i) = \frac{\partial}{\partial q_i} L$$

giving the familiar Lagrange condition

$$\delta \frac{\partial}{\partial(dq_i)} L = \frac{\partial}{\partial(q_i)} L.$$

Given that δ is assumed to be involutive, this is not expressing anything other than that we are looking at a Lie derivative of a pullback from M.

The right side can be manipulated by both adding and then subtracting the term $\sum_i \delta(q_i) d' p_i$. An organized way of doing that is to use Cartan's equation $\delta = d \circ i_\delta + i_\delta \circ d$. Thus for general $\eta\omega = \sum p_i d' q_i$ we obtain

$$\delta\eta\omega = i_\delta(\sum_i d' p_i \wedge d' q_i) + d' \sum p_i \delta(q_i).$$

The first term is the contraction of an alternating form, of course, and the second term is the differential of a quantity $\sum_i p_d \delta(q_i)$ sometimes interpreted as twice the kinetic energy.

The point is that since we assume this is an expression for ω and that ω is closed, so is the first term on the right side. Locally writing this dH, the fact that i_δ is nilpotent of order two forces H to be invariant under δ.[21] The identity between the coefficients of dH and the coefficients of

$$i_\delta \sum_i dp_i \wedge dq_i = \sum \delta(p_i)d'q_i - \delta(q_i)d'p_i$$

is the Hamilton equation which is a simple way of writing an ordinary differential equation determining the rate of change of the p_i and q_i. Note that in our current situation considering δ to be involutive, $\delta(q_i)$ is anyway nothing but dq_i viewed as a section of $\mathcal{O}_N$ and the meaningful information in such an attempt would only be that $\frac{\partial}{\partial q_i}H = \delta(p_i)$.

[21] If H were interpreted as minus the total energy, then the whole right side would be interpreted as the deRham differential on the tangent bundle N of minus the total energy plus twice the kinetic energy, or the difference between kinetic and potential energy, however we'll discuss such notions later on.

As a practical matter, the Hamiltonian approach to solving physical problems can nicely be summarized like this, then. To choose any one-form of your choice pulled back from M, which we locally write if we wish $\sum_i p_i d' q_i$ and whose Lie derivative ω is closed. Then if one manages to describe the function $\sum p_i \delta(q_i)$, which is ordinarily considered to be twice the kinetic energy, one is done. Because the difference

$$\delta \sum_i p_i d' q_i - d' \sum p_i \delta(q_i)$$

is again closed; locally writing this as dH for a function H on the tangent bundle, one then has

$$\delta(p_i) = \partial/\partial q_i H.$$

That is, the rate of change of the p_i are determined by the spatial gradient of H, regardless of however one has chooses the coordinates. The p_i are functions on N and as long as $dp_1 \wedge ... \wedge dp_n$ is nondegenerate as an n form on the tangent space fibers, the dq_i can be expressed in terms of these and the rates of change of the dq_i obtained by the appropriate Jacobian determinant.

This is an appealling approach to physics, due to Hamilton, it does not depend on any notion of energy or of conservation of energy, nor on the interpretation of the function $\sum p_i \delta q_i$ as twice the kinetic energy. The two terms $d'H$ and $d' \sum p_i \delta q_i$ are the two terms of the Cartan equation for $\delta \eta \omega$. In cases when it is possible to identify the tangent and cotangent bundle with each other one sometimes takes for p_i the function $\partial/\partial q_i$ and then $\sum p_i d' q_i$ is the natural one form, though the possibility of doing this relates to one of the problems that a relativistic approach would need to deal with.

Local Legendre transformations

Whenever we manage to furnish a path in the manifold M with a one-form of M along its length, we have lifted that path into the cotangent bundle in such a way that whatever one-form we were looking at has been transformed into the universal form β.

For a one-form ω which is not on M but on the tangent bundle N, the form $\eta\ \omega$ is not by any means natural or universal, and yet it is a section of the pullback from M of one-forms on M. When we calculate an integral of $\eta\ \omega$ along a path, thinking of the path as lying in one section of the tangent bundle, the section is isomorphic to an open subset of M itself, and $\eta\ \omega$ corresponds to a one-form on the open subset. Then that same section, since it is furnished with a one-form, can be interpreted as being a section of the cotangent bundle, and the form $\eta\ \omega$ in now agrees perfectly with the universal one form.

If $q_1, ..., q_n$ are smooth coordinate functions on M then there are uniquely defined smooth functions $p_1, ..., p_n$ on the tangent bundle N such that $\eta\ \omega = \sum p_i dq_i$. Wherever the alternating n-form $dp_1\ dp_2\ ...\ dp_n$ restricts nonzero to fibers of $N \to M$ the p_i and q_i are 'canonical local coordinates' in which Lagrange's equation, for any vector field on N which is a priori assumed to be involutive, is the same[22] as Hamilton's equation.

[22] If we think of a tangent vector as assigning a value to each q_i and a coefficient $\dot{q}_i$ to each $\partial/\partial q_i$ then when locally $\omega = dL$ we have $p_i = \partial/\partial \dot{q}_i L$. Setting $H = L - j\ d\ L = L - \epsilon\ L = L - \sum_i \dot{q}_i \partial/\partial \dot{q}_i L$ gives $dH = \sum_i (\partial/\partial q_i\ L) dq_i - \dot{q}_i d(\partial/\partial \dot{q}_i\ L)$ while $i\ d\ \eta\ \omega = \sum_i \delta(\partial/\partial \dot{q}_i L) dq_i - \delta(q_i) d(\partial/\partial \dot{q}_i L)$. The Lagrange equation says that these are equal as they are written while Hamilton's equation says that $dH = i\ d\ \eta\ \omega$ in the canonical coordinates p_i, q_i.

Poisson brackets

Let M be a manifold with closed two-form ϕ preserved by vector fields δ and τ. This is the same as saying that the dual one-forms $i_\delta\ \phi$ and $i_\tau\ \phi$ are closed. The condition for δ to preserve $i_\tau\ \phi$ is $0 = i_\delta\ d\ i_\tau\ \phi + d\ i_\delta\ i_\tau\ \phi$ with the first term zero. The second term is d of the evaluation of δ and τ under ϕ, it also equals $i_{[\delta,\ \tau]}\ \phi$. Either interpretation shows it's antisymmetric under interchanging δ with τ, giving a sort of reciprocity:

9. Proposition. δ preserves the singular foliation by $i_\tau\ \phi$ if and only if τ preserves the singular foliation by $i_\delta\ \phi$.

Fix now a choice of δ. The τ such that $i_\tau\ \phi$ is closed and which preserve $i_\delta\ \phi$ comprise a Lie algebra L. Given two such τ, τ' the Lie bracket $[\tau,\ \tau']$ still preserves $i_\delta\ \phi$ and $i_{[\tau,\tau']}\ \phi = d\ i_\tau\ i_{\tau'}\ \phi$, is not only closed but exact.

L maps to the kernel of

$$i_\delta : \Omega_M \to \mathcal{O}_M/k.$$

If ϕ is nondegenerate L maps isomorphically onto the closed global sections of the kernel, giving them a Lie algebra structure known as *Poisson bracket*. The foliation by orbits of δ is an intersection of codimension-one foliations if and only if the kernel sheaf has the property that it is the reflexivication of the span of its closed global sections. When this is the case, the foliation is determined by any set of Lie algebra generators.

If we take M to be a tangent bundle, and δ to be involutive with contracting map (2) equivariant for the scalar action and Lagrangian for a one-form ω for which $d\ \eta\ \omega$ is nondegenerate, and if the Lie algebra generators contract $d\ \eta\ \omega$ to exact forms dH then to say that the Lie algebra generates the kernel up to a reflexivication is to say that the H are a complete set of 'classical integrals.'

10. Proposition. Under these conditions, after enlarging the set of H by finitely many iterated Poisson brackets, the equations $H =$ *constant* set-theoretically define every orbit of δ as an intersection of hypersurfaces.

The Lie algebra includes δ which we chose initially, and the classical integral corresponding to δ itself is sometimes called 'energy.'

Discussion

Compactifying N in $Proj$ of the symmetric algebra sheaf of $\Omega_M \oplus \mathcal{O}_M$ gives the compactification of each fiber of $N \to M$ as the complement of a hyperplane in projective space. If M is projective, the divisor at infinity is a copy of the projectivized tangent bundle of M. Some integer multiple of this is a canonical divisor K of the foliation which underlies the chosen involutive vector field δ. The divisor K need not be preserved by the foliation, but for any divisor E which is preserved by the foliation there is a map

$$\Omega_{\overline{N}}(log\ E) \to \mathcal{O}_N(K)$$

whose cokernel is supported on the singular locus of the foliation, which has codimension at least two if M and therefore N is normal. The global sections of the kernel consists entirely of closed one-forms by Deligne's theorem, and any rank one coherent subsheaf of the kernel with two k-linearly independent global sections describes a foliation by the fibers of a rational function by Bogomolov's theorem.

An initial question

The cotangent bundle is admittedly a very appealing object; we mentioned how there any oriented compact surface with boundary has a natural flux which can be found by integrating the canonical form around the oriented boundary.

But a difficulty with the approach of first defining the tangent bundle, where velocity vectors exist, and then defining the different cotangent bundle, where a universal way of measuring velocity vectors exists, is that even in the first thought experiments about relativity, one imagines two people each in his own spaceship, with each one person perceiving the velocity vector of the other.

Maybe it would be better if changing point of view between one observer and the other incorporated the duality between the tangent space and the cotangent space. That is, what is the tangent space for one of the two observers might actually equal the cotangent space for the other.

For example, the difficulty with constructing a global Legendre transformation ought to be related to the question of the event horizons of two different observers who are attempting to relate to one another.

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