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On Resolving Vector Fields

John Atwell Moody

Mathematics Institute, Warwick University, Coventry, CV4 7AL, England

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0. DISCUSSION

For V a singular affine irreducible variety over a field k, and $\mathscr{D}$ an $\mathscr{O}_V$-module of k-linear derivations, we wish to address the question whether there is a blowup $\tilde{V}$ of V such that the subsheaf of the constant sheaf of rational functions

$$\mathscr{O}_{\tilde{V}} \cdot \mathscr{D}(\mathscr{O}_{\tilde{V}})$$

is a locally free coherent sheaf on $\tilde{V}$.

At one extreme is the case $\mathscr{D} = \mathscr{O}_V \cdot \delta$ for $\delta \in \mathscr{D}$ a fixed derivation. In this case, the question is equivalent to whether the vector field δ lifts to a nonsingular one-dimensional foliation on $\tilde{V}$. Significant work has been done on the question of reduction of singularities of vector fields in this situation (see [1] and its sequels).

At the other extreme, assuming V is normal, for any nonsingular blowup $Bl_I(V) = \tilde{V}$ there does exist a $\mathscr{D} \subset \mathscr{D}er_k(\mathscr{O}_V, \mathscr{O}_V)$ such that $\mathscr{O}_{\tilde{V}} \cdot \mathscr{D}(\mathscr{O}_{\tilde{V}})$ is locally free of rank one.

In both these cases, and for general $\mathscr{D}$, the problem is equivalent to finding an ideal $I \subset \mathscr{O}_V$ so that, once modified in a certain way to produce a new ideal $\mathscr{J}(I)$, it must happen that as a fractional ideal $\mathscr{J}(I)$ is a divisor of a power of I:

$$\exists K\ \exists \alpha\ K\mathscr{J}(I) = I^{\alpha}.$$

For purposes of discussion, let's call this the "strict condition." From the previous paragraph, the problem of finding an ideal I satisfying the strict condition for at least some choice of $\mathscr{D}$ is analogous to, but easier than, the problem of resolving the singularities of V. Moreover, understanding either problem well should enlighten the other.

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The fact that I occurs on both sides of the equation reflects possible twisting in the sheaf $\mathscr{O}_{\tilde{V}} \cdot \mathscr{D}(\mathscr{O}_{\tilde{V}})$.

Some of these issues will be discussed in a sequel. In this note, the aim is to replace the strict condition with something a little more manageable. In Theorem 10 of Section 3 we describe a second, more relaxed condition on an ideal J, but such that the existence of an ideal J satisfying the relaxed condition still implies the existence of an ideal I satisfying the strict condition. One of the features of the relaxed condition is that it is an inclusion of ideals, rather than an equality. During the proof, the auxiliary ideal K is eliminated from the equation.

The new condition comes furnished with a rule which, from any solution J produces a solution I of the strict condition, and vice versa.

1. INTRODUCTION: RESOLUTION OF $\mathbb{G}_m$-ACTIONS

In this introductory section we'll illustrate what the calculation says in a special case. In Section 2, all the fundamentals about derivations and blowing up will be developed. None of Section 2 is deep; however, basic facts, such as the fact that one blowup dominates another if and only if one ideal is a divisor of a power of the other, do not seem to occur elsewhere in the literature. In Section 3, the main theorem will be stated in general and proven.

For this section, let V be an irreducible affine variety over k a field of characteristic zero, and suppose V is furnished with a $\mathbb{G}_m$-action. Let's look for an equivariant map $V \to \mathbb{A}^n$ such that up on $\tilde{V} \to V$, the least blowup where the associated map to projective space

$$\begin{array}{ccc} \tilde{V} & \to & \mathbb{P}^{n-1} \\ \downarrow & & \\ V & \to & \mathbb{A}^n \end{array}$$

becomes well-defined, the natural vector field on V lifts to a nonsingular foliation. In this case the coordinate ring of V is a graded k-algebra $\cdots + R_{-2} + R_{-1} + R_0 + R_1 + R_2 + \cdots$ and we are asking whether there is a homogeneous ideal I whose blowup resolves the vector field

$$\begin{aligned} &\delta(\cdots f_{-2} + f_{-1} + f_0 + f_1 + f_2 + f_3 \cdots) \\ &\qquad = \cdots -2f_{-2} - f_{-1} + 0 + f_1 + 2f_2 + 3f_3 \cdots . \end{aligned}$$

We can deduce from the main theorem in Section 3 that for any graded subspace $J \subset R$, we have the following:

THEOREM (Special Case). (i) *Suppose condition* (*) *holds for the set of homogeneous elements of* J. *Then blowing up the ideal* I *generated by all products of three homogeneous elements* ABC *of* J *which are not all three of equal degree resolves the vector field.*

(ii) *Conversely, if there is any homogeneous ideal* I *which resolves the vector field, any sufficiently high power* $J = I^N$ *satisfies condition* (*).

Therefore, the existence of a graded subspace $J \subset R$ satisfying condition (*) is equivalent to the possibility of resolving the vector field by blowing up a homogeneous ideal.

Below is condition (*) in this special case. Note when there exists any graded subspace $J/$subset R satisfying this condition, then there must exist another such choice of J which is a homogeneous ideal, and yet another such choice of J which is a finite-dimensional vector space.

However, of course, J is never concentrated in a single degree unless $I = 0$, in which case the blowup is undefined.

Condition (*). Firstly, any product $x = ABCDEF$ of six homogeneous elements of J such that some subset of four letters neither has a three-of-a-kind (three of the same degree) nor two two-of-a-kind's must have the property that the same element x lies in the ideal generated by all products $HIJKLM$ of six homogeneous elements of J with no four-of-a-kind. Secondly, any product $x = ABCDEFG$ of seven homogeneous elements of R such that the first six A, B, C, D, E, and F lie in J and G has nonzero degree, and such that some subset of four of the first six has no three-of-a-kind (but two two-of-a-kind's are allowed), must have the property that the same element x lies in the ideal generated by all products $HIJKLM$ of six homogeneous elements of J with no four-of-a-kind.

It is worth explaining how Condition (*) here is deduced from the more general condition of Section 3. The products $ABCD$ of homogeneous elements of J with no three-of-a-kind or two two-of-a-kind's form a set of generators of $\mathscr{M}_{\mathscr{D}}\mathscr{M}_{\mathscr{D}}(J)$, where $\mathscr{D}$ is the principal module generated by δ. Therefore the elements x described in the first part of the condition span $J^2\mathscr{M}_{\mathscr{D}}\mathscr{M}_{\mathscr{D}}(J)$. The elements x described in the second part span $J^2(\mathscr{M}_{\mathscr{D}}(J))^2\mathscr{D}(\mathscr{O}_V)$. The requirement is that any such element must lie in $(\mathscr{M}_{\mathscr{D}}(J))^3$, and indeed, this ideal is spanned by products of six homogeneous elements of J with no four degrees of a kind.

2. FUNDAMENTALS

1. LEMMA. *If* $I = (f_1, \ldots, f_m)$ *is an ideal and* $N \geq 1$ *then* $I^{(m-1)N}(f_1^N, \ldots, f_M^N) = I^{mN}$. *Likewise if* $I = A + B$ *then* $I^N(A^N + B^N) = I^{2N}$.

2. THEOREM. *Let* $I = (f_1, \ldots, f_m)$ *and* J *be ideals of an integral domain* R. *The following are equivalent*:

(i) *For some* β *and some fractional ideal* L,

$$f_1^\beta, \ldots, f_m^\beta \in JL \subset I^\beta.$$

(ii) *The pullback of* J *to a sheaf of ideals on* $Bl_I(\operatorname{Spec} R)$ *is locally principal.*

(iii) *There is a map over* Spec R

$$Bl_I(\operatorname{Spec} R) \to Bl_J(\operatorname{Spec} R).$$

(iv) *As a fractional ideal,* J *is a divisor of* I^α *for some* α.

Proof. Suppose (i), so for some β and some fractional ideal L,

$$f_1^\beta, \ldots, f_m^\beta \in JL \subset I^\beta.$$

By the lemma,

$$\left(f_1^\beta, \ldots, f_m^\beta\right) I^{\beta(m-1)} = I^{\beta m}$$

so multiplying through the previous equation by $I^{\beta(m-1)}$ we have

$$I^{\beta m} = JLI^{\beta(m-1)}.$$

Setting $\alpha = \beta m$ we have (iv). Conversely, (iv) directly implies (i).

Next let's show (i) implies (ii). Suppose (i). Let us calculate this pullback of J. We may cover $Bl_I(V)$ by coordinate charts

$$U_i = \operatorname{Spec}(A_i)$$

for

$$A_i = \bigcup_{j=0}^{\infty} (I/f_i)^j.$$

Since JL contains f_i^α, the ideal $J \cdot (L/f_i^\alpha) \cdot A_i \subset A_i$ contains 1. This implies the ideal JA_i is invertible, i.e., locally free. The implication (ii) implies (i) is obtained by reversing this argument, using the fact that a locally free sheaf is locally free on every coordinate patch. Finally, the equivalence between (i) and (iii) comes from the fact that (i) is an algebraic reformulation of (iii).

Let $(f_1, \dots, f_m) = I \subset \mathscr{O}_V$ be an ideal and let $\mathscr{D}$ be an $\mathscr{O}_V$-module of k-linear derivations of $\mathscr{O}_V$.

3. DEFINITION. Let $\mathscr{J}_{\mathscr{D}}(I)$ be the ideal generated by the $a\delta(b) - b\delta(a)$ for $a, b \in I$, $\delta \in \mathscr{D}$.

Also,

4. DEFINITION. If $I = (f_1, \dots, f_n)$, let $\mathscr{M}_{\mathscr{D}}(f_1, \dots, f_n)$ be the ideal of $\mathscr{O}_V$ generated by the $f_i\delta(f_j) - f_j\delta(f_i)$ for $\delta \in \mathscr{D}$.

5. THEOREM. $\mathscr{J}_{\mathscr{D}}(I) = I^2\mathscr{D}(\mathscr{O}_V) + \mathscr{M}_{\mathscr{D}}(f_1, \dots, f_n)$.

Proof. Clearly $\mathscr{M}_{\mathscr{D}}(f_1, \dots, f_n) \subset \mathscr{J}_{\mathscr{D}}(I)$, the right side containing each generator of the left by definition. Also, $\mathscr{J}_{\mathscr{D}}(I) \subset I^2\mathscr{D}(\mathscr{O}_V) + \mathscr{M}_{\mathscr{D}}(f_1, \dots, f_n)$ because each element of the left side is a sum of elements of the form $af_i\delta(bf_j) - \delta(af_i)bf_j = f_if_j(a\delta(b) - b\delta(a)) + ab(f_i\delta(f_j) - \delta(f_i)f_j)$. It remains to prove $I^2\mathscr{D}(\mathscr{O}_V) \subset J_{\mathscr{D}}(I)$. Thus take a typical generator $f_if_j\delta(x_k)$. Let

$$A = \delta(x_kf_j)f_i - x_kf_j\delta(f_i) \in J_\delta(I).$$

We have

$$A = \delta(x_k)f_if_j + x_k\delta(f_j)f_i - x_kf_j\delta(f_i).$$

Let $B = \delta(f_j)f_i - f_j\delta(f_i) \in \mathscr{J}_\delta(I)$. Now $f_if_j\delta(x_k) = A - x_kB \in \mathscr{J}_\delta(I)$.

Let $(f_1, \dots, f_m)$ be a sequence of generators of an ideal $I \subset \mathscr{O}_V$. Let $x_1, \dots, x_n$ be a sequence of k-algebra generators of $\mathscr{O}_V$.

The proof of Theorem 5 shows that

6. THEOREM. *If a sequence $(f_1, \dots, f_n)$ of generators of I is enlarged by appending all products of the f_i with a system $(x_1, \dots, x_m)$ of k-algebra generators of $\mathscr{O}$, the new sequence $(f_1, \dots, f_N)$ has the property that $\mathscr{J}_{\mathscr{D}}(I) = \mathscr{M}_{\mathscr{D}}(f_1, \dots, f_N)$.*

7. LEMMA. *The pullback of $\mathscr{J}_{\mathscr{D}}(I)$ to $Bl_I(V)$ is the sheaf of ideals which, on the coordinate chart* Spec(A_i), *agrees with the ideal generated by the $f_i^2\delta(x_h)$ and the $f_i^2\delta(f_j/f_i)$ (note i is now fixed).*

Proof. Let π be the structure map of the blowup. For each i let

$$A_i = \bigcup_{N=0}^{\infty} \frac{I^N}{f_i^N}.$$

Let $U_i = \text{Spec}(A_i)$, so the U_i form an open cover of $\tilde{V} = Bl_I(V)$. We have for each i,

$$\pi^*\mathscr{J}_{\mathscr{D}}(I_{\mathscr{D}})(U_i)$$

is the ideal of U_i with generators

$$\left(f_l f_j \tau(x_u), f_l \tau(f_j) - f_j \tau(f_l)\right),$$

where u, l, and j range over the appropriate numbers, the x_u range over any set of k-algebra generators of $\mathscr{O}_V$, and τ ranges over the elements of $\mathscr{D}$. For $r, s \neq i$, we have

$$f_r \tau(f_s) - f_s \tau(f_r) = \frac{f_r}{f_i} \cdot (f_i \tau(f_s) - f_s \tau(f_i)) - \frac{f_s}{f_i} \cdot (f_i \tau(f_r) - f_r \tau(f_i)).$$

Recalling that i is fixed, we now may remove the generators of the second type for which $l \neq i$. Also, for r, s not both equal to i we have

$$f_r f_s \tau(x_u) = f_i^2 \cdot \frac{f_r}{f_i} \frac{f_s}{f_i} \tau(x_u).$$

We may then remove the generators of the first type except when $r = s = i$. Our ideal is now generated by the

$$\left(f_i^2 \tau(x_u), f_i \tau(f_s) - f_s \tau(f_i)\right).$$

The next theorem explains why one has become interested in the ideal $\mathscr{I}_{\mathscr{D}}(I)$. Let $\tilde{V} = Bl_I(V)$. Then

8. THEOREM. *$\pi^*(\mathscr{I}_{\mathscr{D}} I) = \mathscr{O}_{\tilde{V}} \mathscr{D}(\mathscr{O}_{\tilde{V}})(-2E)$, where E is the exceptional divisor associated to blowing up I.*

Proof. This follows immediately from the previous lemma, since the x_h and the f_j/f_i are a system of k-algebra generators of the ith coordinate ring A_i. We merely apply δ and multiply by the square of the local generator of E, which is f_i.

9. THEOREM. *If the characteristic of k is not equal to 2 and $A = (f_1, \dots, f_n)$, $B = (g_1, \dots, g_m) \subset \mathscr{O}_V$ then*

$$M_{\mathscr{D}}((f_1, \dots, f_n)(g_1, \dots, g_m)) = A^2 M_{\mathscr{D}}(g_1, \dots, g_m) + B^2 M_{\mathscr{D}}(f_1, \dots, f_n)$$

and

$$\mathscr{I}_{\mathscr{D}}(AB) = A^2 \mathscr{I}_{\mathscr{D}}(B) + B^2 \mathscr{I}_{\mathscr{D}}(A).$$

Here the product $(f_1, \dots, f_m)(g_1, \dots, g_n)$ denotes the sequence of all pairwise products.

Proof. By Theorem 6, it is possible to choose the sequences of generators large enough so that $\mathscr{M}_{\mathscr{D}}(f_1, \dots, f_n) = \mathscr{I}_{\mathscr{D}}(A)$, etc., so the first formula implies the second. To prove the first formula, a typical distinguished generator of $M((f_1,\dots,f_n)(g_1,\dots,g_m))$ is

$$f_i g_j \delta(f_k g_l) - f_k g_l \delta(f_i g_j)$$
$$= f_i f_k\big(g_j\delta(g_l) - g_l\delta(g_j)\big) + g_l g_j\big(f_i\delta(f_k) - f_k\delta(f_i)\big),$$

which is in $A^2M(B) + B^2M(A)$. Conversely, to obtain a typical generator, say,

$$f_i f_k\big(g_j\delta(g_l) - g_l\delta(g_j)\big)$$

of the sum, we write

$$\tfrac{1}{2}\big(f_i g_j \delta(f_k g_l) - f_k g_l \delta(f_i g_j) + f_k g_j \delta(f_i g_l) - f_i g_l \delta(f_k g_j)\big)$$
$$= \tfrac{1}{2}\big(f_i f_k\big(g_j\delta(g_l) - g_l\delta(g_j)\big) + g_l g_j\big(f_i\delta(f_k) - f_k\delta(f_i)\big)$$
$$+ f_k f_i\big(g_j\delta(g_l) - g_l\delta(g_j)\big) - g_l g_j\big(f_i\delta(f_k) - f_k\delta(f_i)\big)\big)$$
$$= f_i f_k\big(g_j\delta(g_l) - g_l\delta(g_j)\big).$$

The first line is clearly an element of the ideal $\mathscr{M}_{\mathscr{D}}((f_1,\dots,f_n)(g_1,\dots,g_m))$.

3. THE THEOREM

For this section k is allowed to have any characteristic except two. To make sense of the theorem, it is necessary to view $\mathscr{M}_{\mathscr{D}}(f_1,\dots,f_m)$ sometimes as an ideal with a particular generating sequence (so $\mathscr{M}_{\mathscr{D}}$ can be applied again). One may take the sequence of generators which is given in the definition of that ideal, letting δ range over an arbitrary $\mathscr{O}_V$-module generating set of $\mathscr{D}$.

Let $\mathscr{D}$ be any $\mathscr{O}_V$-module of k-linear derivations. For any sequence $(f_1,\dots,f_m) \subset \mathscr{O}_V$, letting J be the ideal $J = (f_1,\dots,f_m)$, let's say the sequence $(f_1,\dots,f_m)$ satisfies Condition (*) if

$$J^2\Big(\mathscr{M}_{\mathscr{D}}\mathscr{M}_{\mathscr{D}}(f_1,\dots,f_m) + \big(\mathscr{M}_{\mathscr{D}}(f_1,\dots,f_m)\big)^2\mathscr{D}(\mathscr{O}_V)\Big)$$
$$\subset \big(\mathscr{M}_{\mathscr{D}}(f_1,\dots,f_m)\big)^3.$$

10. THEOREM. (i) *Let $I \subset \mathscr{O}_V$ be any ideal such that the blowup $\tilde{V} = Bl_I(V)$ satisfies the property that*

$$O_{\tilde{V}}\mathscr{D}(\mathscr{O}_{\tilde{V}})$$

is locally free. Then for $N \gg 0$ *the ideal* $J = I^N$ *has a sequence of generators which satisfies Condition* (*) (*precisely, any sequence of generators of* J *together with their products with a system of k-algebra generators of* $\mathscr{O}$ *will do*).

(ii) *Let* $(f_1, \ldots, f_m) \subset \mathscr{O}_V$ *be any sequence of elements satisfying condition* (*), *and write* $J = (f_1, \ldots, f_m)$ *the associated ideal. Then the ideal* $I = J \cdot \mathscr{M}_{\mathscr{D}}(f_1, \ldots, f_m)$ *is such that the blowup* $\tilde{V} = Bl_I(V)$ *satisfies the property that*

$$\mathscr{O}_{\tilde{V}} \mathscr{D}(\mathscr{O}_{\tilde{V}})$$

is locally free.

Remark. Everything here remains true if the sequences of functions $(f_1, \ldots, f_m)$ are allowed to be infinite. This is important for the application in the Introduction, because there we take as our generating functions the full set of homogeneous elements of an ideal. We may as well say m is allowed to be any cardinal number.

Proof. Suppose I is such that the blowup $\tilde{V} = Bl_I(V)$ satisfies the property that

$$\mathscr{O}_{\tilde{V}} \mathscr{D}(\mathscr{O}_{\tilde{V}})$$

is locally free. This is of course the same as saying that

$$\mathscr{O}_{\tilde{V}} \mathscr{D}(\mathscr{O}_{\tilde{V}})(-2E)$$

is locally free. By Theorem 8,

$$\mathscr{O}_{\tilde{V}} \mathscr{D}(\mathscr{O}_{\tilde{V}})(-2E) \cong \pi^*(\mathscr{J}_{\mathscr{D}}(I)).$$

By Theorem 2, this is locally free if and only if there is a number α and an ideal K such that

$$K \mathscr{J}_{\mathscr{D}}(I) = I^{\alpha}.$$

Now let's calculate $K \mathscr{J}_{\mathscr{D}}(I^{\alpha+1})$. Using Theorem 9 plus induction, we have

$$K \mathscr{J}_{\mathscr{D}}(I^{\alpha+1}) = K I^{2\alpha} \mathscr{J}_{\mathscr{D}}(I).$$

This is then equal to

$$= I^{2\alpha} I^{\alpha} = I^{3\alpha},$$

and it follows that

$$(KI^3) \mathscr{J}_{\mathscr{D}}(I^{\alpha+1}) = (I^{\alpha+1})^3.$$

Let us set $N = 2\alpha + 2$, $J = I^N$, and calculate

$$J^2\mathscr{I}_{\mathscr{D}}\mathscr{I}_{\mathscr{D}}J = I^{4\alpha+4}\mathscr{I}_{\mathscr{D}}\mathscr{I}_{\mathscr{D}}(I^{2\alpha+2}).$$

Writing $I^{2\alpha+2} = I^{\alpha+1}I^{\alpha+1}$ and using Theorem 9, we have

$$\begin{aligned}\mathscr{I}_{\mathscr{D}}(I^{2\alpha+2}) &= \mathscr{I}_{\mathscr{D}}(I^{\alpha+1}I^{\alpha+1})\\ &= I^{2\alpha+2}\mathscr{I}_{\mathscr{D}}(I^{\alpha+1}).\end{aligned}$$

Substituting, we find

$$J^2\mathscr{I}_{\mathscr{D}}\mathscr{I}_{\mathscr{D}}(J) = \left(I^{4\alpha+4}\mathscr{I}_{\mathscr{D}}\left(I^{2\alpha+2}\mathscr{I}_D I^{\alpha+1}\right)\right).$$

Again using Theorem 9, we have the formula

$$\mathscr{I}_{\mathscr{D}}\left(I^{2\alpha+2}\mathscr{I}_{\mathscr{D}}I^{\alpha+1}\right) = I^{4\alpha+4}\mathscr{I}_{\mathscr{D}}\mathscr{I}_{\mathscr{D}}I^{\alpha+1} + \left(\mathscr{I}_{\mathscr{D}}I^{\alpha+1}\right)^2\mathscr{I}_{\mathscr{D}}I^{2\alpha+2},$$

from which we obtain

$$\begin{aligned}J^2\mathscr{I}_{\mathscr{D}}\mathscr{I}_{\mathscr{D}}J &= I^{8\alpha+8}\mathscr{I}_{\mathscr{D}}\mathscr{I}_{\mathscr{D}}I^{\alpha+1} + I^{6\alpha+6}\left(\mathscr{I}_{\mathscr{D}}I^{\alpha+1}\right)^3\\ &= \left(KI^3\mathscr{I}_{\mathscr{D}}I^{\alpha+1}\right)^2\cdot\left(I^{2\alpha+2}\mathscr{I}_{\mathscr{D}}\mathscr{I}_{\mathscr{D}}I^{\alpha+1} + \left(\mathscr{I}_{\mathscr{D}}I^{\alpha+1}\right)^3\right).\end{aligned}$$

Now, another application of Theorem 9 shows us that

$$(KI^3)^2\mathscr{I}_{\mathscr{D}}\mathscr{I}_{\mathscr{D}}I^{\alpha+1} + \left(\mathscr{I}_{\mathscr{D}}I^{\alpha+1}\right)^2\mathscr{I}_{\mathscr{D}}(KI^3) = \mathscr{I}_{\mathscr{D}}\left(KI^3\mathscr{I}_{\mathscr{D}}(I^{\alpha+1})\right),$$

from which we obtain the inclusion

$$(KI^3)^2\mathscr{I}_{\mathscr{D}}\mathscr{I}_{\mathscr{D}}I^{\alpha+1} \subset \mathscr{I}_{\mathscr{D}}\left(KI^3\mathscr{I}_{\mathscr{D}}I^{a+1}\right).$$

This gives us

$$J^2\mathscr{I}_{\mathscr{D}}\mathscr{I}_{\mathscr{D}}(J) \subset \left(\mathscr{I}_{\mathscr{D}}I^{\alpha+1}\right)^2\cdot I^{2\alpha+2}\mathscr{I}_{\mathscr{D}}\left(KI^3\mathscr{I}_{\mathscr{D}}I^{\alpha+1}\right) + (KI^3)^2\left(\mathscr{I}_{\mathscr{D}}I^{\alpha+1}\right)^5.$$

Now we are in a position to eliminate K. We find

$$J^2\mathscr{I}_{\mathscr{D}}\mathscr{I}_{\mathscr{D}}J \subset \left(\mathscr{I}_{\mathscr{D}}I^{\alpha+1}\right)^2 I^{2\alpha+2}\mathscr{I}_{\mathscr{D}}(I^{3\alpha+3}) + (I^{\alpha+1})^6\left(\mathscr{I}_{\mathscr{D}}I^{\alpha+1}\right)^3.$$

Applying Theorem 9 to the first term we see it is equal to the second, so we finally conclude

$$J^2\mathscr{I}_{\mathscr{D}}\mathscr{I}_{\mathscr{D}}(J) \subset (I^{\alpha+1})^6\left(\mathscr{I}_{\mathscr{D}}I^{\alpha+1}\right)^3 = \mathscr{I}_{\mathscr{D}}(J)^3.$$

Now, with respect to any sequence of generators of $\mathcal{J}_{\mathcal{D}}(J)$ we may rewrite the left side of the equation according to Theorem 5:

$$J^2\Big(\mathcal{M}_{\mathcal{D}}(\mathcal{J}_{\mathcal{D}}J) + (\mathcal{J}_{\mathcal{D}}J)^2\mathcal{D}(\mathcal{O}_V)\Big) = J^2\mathcal{J}_{\mathcal{D}}\mathcal{J}_{\mathcal{D}}J.$$

To produce the sequence of generators implicit in the statement of the theorem, begin with any sequence of generators $(f_1, \ldots, f_m)$ of J, and enlarge it according to Theorem 6, so that $\mathcal{J}_{\mathcal{D}}(J) = \mathcal{M}_{\mathcal{D}}(f_1, \ldots, f_m)$. Also choose a system of $\mathcal{O}$-module generators $\delta \in \mathcal{D}$, so the right side of the equation above is generated by the $f_i\delta(f) - f_j\delta(f_i)$. With respect to this system of generators it makes sense to apply $\mathcal{M}_{\mathcal{D}}$ again, using Theorem 5 we finally obtain the formula

$$J^2\Big(\mathcal{M}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}}(f_1, \ldots, f_m) + \big(\mathcal{M}_{\mathcal{D}}(f_1, \ldots, f_m)\big)^2\mathcal{D}(\mathcal{O}_V)\Big) \subset \mathcal{M}_{\mathcal{D}}(f_1, \ldots, f_m)^3,$$

which proves Condition (*).

Conversely, if this condition holds, let $I = J\mathcal{M}_{\mathcal{D}}(f_1, \ldots, f_m)$. We have

$$\mathcal{J}_{\mathcal{D}}(I) = \mathcal{J}_{\mathcal{D}}\big(J\mathcal{M}_{\mathcal{D}}(f_1, \ldots, f_m)\big).$$

By Theorem 9 and Theorem 5 this is equal to

$$J^2\mathcal{M}_{\mathcal{D}}\mathcal{M}_{\mathcal{D}}(f_1, \ldots, f_m) + \big(\mathcal{M}_{\mathcal{D}}(f_1, \ldots, f_m)\big)^2\mathcal{J}_{\mathcal{D}}J.$$

We can expand $\mathcal{J}_{\mathcal{D}}(J)$ out by Theorem 5

$$\mathcal{J}_{\mathcal{D}}(J) = \mathcal{M}_{\mathcal{D}}(f_1, \ldots, f_m) + J^2\mathcal{D}(\mathcal{O}_V).$$

Substituting, we find

$$\begin{aligned}\mathcal{J}_{\mathcal{D}}(I) = J^2\mathcal{M}_{\mathcal{D}}\big(\mathcal{M}_{\mathcal{D}}(f_1, \ldots, f_m)\big) + \big(\mathcal{M}_{\mathcal{D}}(f_1, \ldots, f_m)\big)^3 \\ + \big(\mathcal{M}_{\mathcal{D}}(f_1, \ldots, f_m)\big)^2 J^2\mathcal{D}(\mathcal{O}_V).\end{aligned}$$

The first term plus the last term are contained in the middle term by Condition (*), and so

$$\mathcal{J}_{\mathcal{D}}(I) = \big(\mathcal{M}_{\mathcal{D}}(f_1, \ldots, f_m)\big)^3.$$

Since

$$I = J\mathcal{M}_{\mathcal{D}}(f_1, \ldots, f_m),$$

we can let $K = J^3$, and we have

$$K\mathcal{J}_{\mathcal{D}}(I) = I^3.$$

By Theorem 2 this proves that when we blowup I, the ideal $\mathscr{I}_{\mathscr{D}}(I)$ pulls back to a locally free sheaf, and then by Theorem 8 this proves that

$$\mathscr{O}_{\tilde{V}}\mathscr{D}(\mathscr{O}_{\tilde{V}})(-2E)$$

is locally free. Untwisting the sheaf does not affect local freeness and we are done.

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REFERENCE

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