Indeterminacy of a morphism

A rational morphism of projective varieties gives a torsion free rank one coherent sheaf $\mathcal{F}$ on the source variety, such that the indeterminacy of the morphism is the locus where $\mathcal{F}$ fails to be locally free. We wish to describe a coherent sheaf depending on $\mathcal{F}$ which is supported on the indeterminacy locus.

If we wish to work homologically, we could use the support of the functor $\mathcal{E}xt(\mathcal{F}, -)$. Or, if the source variety is locally factorial, we can use the quotient of the reflexivication of $\mathcal{F}$ modulo $\mathcal{F}$.

The goal of this note is to construct a better object, a coherent sheaf $Ind_X(\mathcal{F})$ on X whose support is the indeterminacy of the morphism, and which restricts by passage to a subsheaf, by a residue map of coherent sheaves $(i^*\mathcal{F})^c\Lambda^c\mathcal{N}_{V/X}Ind_V(i^*\mathcal{F}) \to i^*Ind_X(\mathcal{F})$ for any smooth subvariety $i : V \subset X$ of codimension c. This becomes an embedding if we are more careful with things like reducing modulo torsion and multiplying by a higher power of $i^*\mathcal{F}$ however these details are not very important and are only sketched here.

Suppose $i : V \subset X$ is an embedding of nonsingular varieties. Write c for the codimension of V and d for the dimension so $c + d = n = dim(X)$.

Also let $\mathcal{F}$ be a torsion free coherent sheaf of rank one on X.

There is an exact diagram

$$\begin{array}{ccccccccc}
 & & \mathcal{F}\mathcal{N}_{V/X} & = & \mathcal{F}\mathcal{N}_{V/X} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & i^*(\mathcal{F}\Omega_X) & \to & i^*(\mathcal{P}(\mathcal{F})) & \to & i^*\mathcal{F} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (i^*\mathcal{F})\Omega_V & \to & \mathcal{P}(i^*\mathcal{F}) & \to & i^*\mathcal{F} & \to & 0
\end{array}$$

The locus of indeterminacy of $\mathcal{F}$ is determined by the coherent sheaf which is the highest exterior power of the middle term of the middle row (mod torsion) without i^* applied, modulo the product of the highest exterior powers of the two end terms up to multiplying by a power of $\mathcal{F}$ and reducing modulo torsion. Let's call this $Ind_X(\mathcal{F})$. It is not easily possible to describe a sheaf with the same suppport without using exterior products. Note that we're being a little imprecise since the definition of $Ind_X(\mathcal{F})$ should specify what power of $\mathcal{F}$ we have multiplied by, for now let's just say one fixed very high power.

The analogous locus of indeterminacy of $i^*\mathcal{F}$ is given by the analogous calculation for the lower row. We call this $Ind_Y(i^*\mathcal{F})$.

The left column (with zeroes included at the top and bottom of each sequence too) is locally split, and so the n'th exterior power of $i^*\mathcal{F}\Omega_X$ is the c'th exterior power of $\mathcal{F}\mathcal{N}$ times the d'th exterior power of $(i^*\mathcal{F})\Omega_V$. The elements in the highest exterior power of $i^*\mathcal{P}(\mathcal{F})$ (mod torsion) which are trivial in Ind then are elements in the tensor product of highest exterior powers of three sheaves. We wish to show that a local section of the highest exterior power of $\mathcal{P}(i^*\mathcal{F})$ is trivial in the indeterminacy sheaf of $i^*\mathcal{F}$ on the subvariety V if and only if its image in the highest exterior power of $i^*\mathcal{P}(\mathcal{F})$ is trivial in the indeterminacy sheaf of the variety X.

Often given a map $i : V \to X$ one constructs maps from the pullback of a sheaf on X to a corresponding sheaf on V. By contrast, in the current situation, the map of interest actually goes in the backward direction. Start with the map

$$i^*\Lambda^{n+1}\mathcal{P}(\mathcal{F}) \leftarrow \mathcal{F}^c\Lambda^c\mathcal{N} \otimes \Lambda^{d+1}\mathcal{P}(i^*\mathcal{F})$$

(recall that We implicitly allow multiplying by an arbitrary higher power of $i^*\mathcal{F}$ and reducing mod torsion.) The trivial elements on the left are those in $i^*\mathcal{F}^c\Lambda^c\mathcal{N}i^*\mathcal{F}\Lambda^d((i^*\mathcal{F})\Omega_V) = i^*\mathcal{F}^{c+d+1}\Lambda^c\mathcal{N}\Lambda^d\Omega_V$.

These not only come from elements trivial on the right, they comprise the same exact sheaf. The map of indeterminacy sheaves is then an embedding

$$i^*Ind_X(\mathcal{F}) \leftarrow \mathcal{F}^c\Lambda^c\mathcal{N} \otimes Ind_V(i^*\mathcal{F}).$$

The twisting by $\Lambda^c\mathcal{N}$ has no effect on the support of course, it seems to be a generalization of what occurs with Poincare residues. Also the multiplication by $\mathcal{F}^c$ has no effect on the support.

Just again to clarify, in the definition of $Ind_X(\mathcal{F})$ before reducing modulo the 'trivial' subsheaf we have multiplied by a very high power of $\mathcal{F}$, and also reduced modulo torsion. likewise in the definition of $Ind_V(i^*\mathcal{F})$ we have multiplied by the same high power of

$i^*\mathcal{F}$, and also reduced modulo torsion. The pullback mod torsion of trivial elements on X is identical then to the trivial elements on V, and reducing modulo trivial elements we again have an embedding of now torsion sheaves on V, the larger one is the indeterminacy sheaf on X pulled back to V (now we do not reduce mod torsion) and the other is the indeterminacy sheaf on V, which is a subsheaf.

It is easy to see why the indeterminacy sheaf of V itself should be allowed to be smaller than the whole of the pullback of the indeterminacy sheaf on X. For example, if V is any smooth curve it picks up a zero indeterminacy sheaf, the subsheaf is zero.

Now, consider the case where $\pi : X \to Y$ is a fiber bundle of smooth varieties, with fibers of dimesion d, and we take $\mathcal{F} = \pi^*\pi_*\Lambda^d\Omega^{\otimes i}_{X/Y}/torsion$ for i chosen as in the minimal model program. One would have to check that i can be made finite even though it is a family of varieties. Then $\mathcal{F}$ gives the canonical rational morphism of each fiber $V_y \subset X$ and for each y we have $\mathcal{F}^c Ind(i_y^*\mathcal{F}) \subset i_y^* Ind(\mathcal{F})$. It is a natural question whether we can find a relative sheaf $Ind_{X/Y}$ such that $Ind(i_y^*\mathcal{F}) = i_y^* Ind_{X/Y}$ for all $y \in Y$. Then the support of the relative sheaf would be a variety whose intersection with each V_y gives the indeterminacy of the canonical morphism of V_y.