Hodge Theory sheet

Let M be a complex manifold of dimension n and let $\mathcal{S}$ be the sheaf of smooth real functions on M (just ignoring the complex structure). That is, to any open subset $U \subset M$ the sheaf $\mathcal{S}$ assigns $\mathcal{S}(U) = C^\infty(U, \mathbb{R})$.

If you do not like sheaves, almost nothing will be different if you consider only the ring $\mathcal{S}(M) = C^\infty(M, \mathbb{R})$ of smooth functions on M.

1. Exercise. Show that $\mathcal{S} \otimes_{\mathbb{R}} \mathbb{C}$ is no different than (is isomorphic to) the sheaf of $\mathcal{Y}$ of smooth maps of real manifolds from M to $\mathbb{C}$, rigorously $\mathcal{Y}(U) = C^\infty(U, \mathbb{C})$.

Now, whenever $f : M \to \mathbb{C}$ is a map of smooth *real* manifolds, let's write $\overline{f}$ for the composite $M \xrightarrow{f} \mathbb{C} \xrightarrow{\sigma} \mathbb{C}$ where σ is complex conjugation. If f is holomorphic we'll call $\overline{f}$ *anti-holomorphic.*

2. Exercise. Show that the resulting action $f \mapsto \overline{f}$ is the same as the one induced by the conjugation automorphism of $\mathbb{C}$ acting on $\mathcal{S} \otimes_{\mathbb{R}} \mathbb{C}$ via the second tensor factor.

3. Exercise. Show that $\mathcal{Y}$ contains (a copy of) both the sheaf $\mathcal{O}_M$ of complex holomorphic functions and also the sheaf $\overline{\mathcal{O}}_M$ of complex anti-holomorphic functions and the action of σ interchanges the two factors.

4. Exercise. Show more precisely that $\mathcal{Y}$ contains a copy of the (semilinear) tensor product $\mathcal{O}_M \otimes_{\mathbb{C}} \overline{\mathcal{O}_M}$ and the action of σ on a section of the type $f \otimes \overline{g}$ sends it to $g \otimes \overline{f}$ for f, g holomorphic.

Let Ω be the sheaf of differentials of $\mathcal{Y}$ over $\mathbb{C}$. That is, the sheaf of complex valued one forms on the real manifold underlying M. Define the (obvious) action of complex conjugation on Ω so that a form fdg is sent to $\overline{f}d\overline{g}$.

5. Exercise. Show that there is a $\mathbb{C}$-linear action of the unitary group $U(1) \subset \mathbb{C}^{\times}$ on Ω defined by $u \cdot \; fdg = fd(ug)$ when g is either holomorphic or antiholomorphic. Show that Ω decomposes naturally into two isotypical parts (one where g is taken to be holomorphic and one where g is taken to be anti-holomorphic). Index the isotypical components of Ω according to which tensor power of the standard one dimensional complex representation of $U(1) \subset \mathbb{C}^{\times}$ each is made up of, giving the decomposition

$$\Omega = \Omega^{-1} \oplus \Omega^{+1}.$$

6. Exercise. Show that the (complex) symmetric and exterior powers of Ω have the isotypical decompositions

$$S^j(\Omega) = S^j(\Omega)^{-j} \oplus S^j(\Omega)^{-j+2} \oplus ... \oplus S^j(\Omega)^j$$

$$\Lambda^j(\Omega) = \Lambda^j(\Omega)^{-j} \oplus \Lambda^j(\Omega)^{-j+2} ... \oplus \Lambda^j(\Omega)^j.$$

with

$$\Lambda^j(\Omega)^i = \Lambda^a(\Omega^{-1}) \otimes \Lambda^b(\Omega^{+1})$$

$$S^j(\Omega)^i = S^a(\Omega^{-1}) \otimes S^b(\Omega^{+1})$$

for a, b chosen such that

$$i = b - a$$

$$j = b + a.$$

7. Exercise. Show that for $i = 0$ and $j = 2$ there is a special isomorphism $S^j(\Omega)^i \cong \Lambda^j(\Omega)^i$. In other words that the $U(1)$ invariant subspace of both $S^2(\Omega)$ and of $\Lambda^2(\Omega)$ can be identified with one and the same space $\Omega^{+1} \otimes \Omega^{-1}$.

8. Exercise. Show that the σ invariant subspace of the $U(1)$-invariant space in the previous exercise is the same as the $U(1) \rtimes \langle\sigma\rangle$-invariant subspace. Show that because of the previous exercise this vector space – the $U(1) \rtimes \langle\sigma\rangle$-invariant subspace of either $S^2(\Omega)$ or $\Lambda^2(\Omega)$ – has two different interpretations: either as the differential $(1,1)$ forms on M which are invariant under the σ action (meaning that when written locally as $\sum h_{\alpha,\beta} dz\alpha \wedge d\overline{z_\beta}$ one has $h_{\alpha,\beta} = \overline{h}_{\beta,\alpha}$) or as the real quadratic forms on M which are invariant under the action of $U(1)$.

Note: The Kahler condition is the conjunction of positive definiteness of the quadratic form – so that it is a Riemann metric – along with closedness of the $(1,1)$ form.

9. Exercise. Show that despite being really the same object, if we interpret a σ invariant section of $\Omega^{-1} \otimes \Omega^{+1}$ as being a $(1,1)$ form, its integral over any closed surface in M is *purely imaginary*, whereas, if we interpret the same object as a $U(1)$ invariant quadratic form, its evaluation of the length of any arc in M is *purely real.*

10. Exercise. Invoke the DeRham theorem and the Dolbeault theorem to show that the cohomology of $\Lambda^{\cdot}\Omega$ with respect to $\overline{\partial}$ gives $\oplus_{p,q} H^q(M, \Lambda^p \Omega_M^{holomorphic})$ while the cohomology of $\Lambda^{\cdot}\Omega$ with respect to $\partial + \overline{\partial}$ is $\oplus_n H^n(M, \mathbb{C})$. Show that $\overline{\partial}$ decreases the $U(1)$ degree by one, so for each q, p the vector space $H^q(M, \Lambda^p(\Omega_M^{holomorphic}))$ is a subquotient of $\Lambda^{p+q}(\Omega)^{p-q} \cong \Lambda^p(\Omega^{+1}) \otimes \Lambda^q(\Omega^{-1})$. Is it preserved by $U(1)$? If so, it is isotypic of character degree $p - q$.

11. Exercise. Show that every cohomology class in $H^2(M, \mathbb{C})$ can be represented by a global section

$$\alpha \oplus \beta \oplus \gamma \in \Lambda^2(\Omega^{-1}) \oplus (\Omega^{-1} \otimes \Omega^{+1}) \oplus \Lambda^2(\Omega^{+1})$$

such that

$$\begin{aligned} 0 &= \overline{\partial}\alpha \\ \partial\alpha &= \overline{\partial}\beta \\ \partial\beta &= \overline{\partial}\gamma \\ \partial\gamma &= 0. \end{aligned}$$

Note: When the first equation is satisfied, the the map $\alpha \mapsto \partial\alpha$ describes the "E_1 differential" acting on α in the Frolicher spectral sequence. When the second equation satisfied it means the E_1 differential sends the class of α to zero, and the element $\partial\beta$ is the image of the class of α under the "E_2 differential" etcetera.

12. Exercise. Show that the cohomology class represented by $\alpha \oplus \beta \oplus \gamma$ in the previous exercise comes from a line bundle if and only if it belongs to $H^2(M, 2\pi i\mathbb{Z}) \subset H^2(M, \mathbb{C})$ and there is a $\delta \in \Omega^{-1}$ so that $\overline{\partial}\delta = \alpha$. Here the -1 denotes an isotypical component, not an inverse. (hint: because of the exponential exact sequence the issue is whether the image in $H^2(M, \mathcal{O}_M)$ is zero.)

13. Exercise. Continuing with the previous exercise, show that when there is such a δ, then replacing β by $\beta - \partial\delta$ arranges that $\overline{\partial}\beta = 0$. Conclude that the cohomology class represents a line bundle if and only if it can be represented by such a global section $\alpha \oplus \beta \oplus \gamma$ with $\alpha = 0$ and $\overline{\partial}\beta = 0$. In particular that the class of β so chosen describes an element of $H^1(M, \Omega_M^{holomorphic})/\partial H^1(M, \mathcal{O}_M)$ naturally associated to the same line bundle. The denominator occurs because of non-uniqueness of δ.

14. Exercise. Explain why there might be two line bundles giving different elements of $H^2(M, 2\pi i\mathbb{Z})$ but which give the same element of $H^1(M, \Omega_M^{holomorphic})/\partial H^1(M, \mathcal{O}_M)$, yet that there cannot be if the second ∂ cohomology of $\Lambda^{\cdot}(\Omega_M^{holomorphic})$ is trivial.

15. Exercise. Prove (without using the Kahler assumption) that if the cohomology class of a closed $(1,1)$ form belongs to the lattice $H^2(M, 2\pi i\mathbb{Z}) \subset H^2(M, \mathbb{C})$ then the $(1,1)$ form comes from a line bundle. (hint: you may use previous exercises).

16. Exercise Suppose you are given real linearly independent elements $v_1, ..., v_{2n} \in \mathbb{C}^n$. Show that the quotient of $\mathbb{C}^n$ modulo these translations satisfies the condition of Kodaira's embedding theorem, and is therefore a projective variety, if there is an n by n positive definite Hermitian matrix h such that for each pair of numbers $1 \leq \alpha_1 < \alpha_2 \leq 2n$ the sum

$$\sum_{i,j=1}^{n} h_{ij} \; det \begin{pmatrix} v_{\alpha_1,i} & \overline{v_{\alpha_1,j}} \\ v_{\alpha_2,i} & \overline{v_{\alpha_2,j}} \end{pmatrix}$$

belongs to $2\pi i\mathbb{Z}$. (hint: when $h_{i,j} \in \mathbb{C}$ are constant, the form $\sum h_{i,j} dz_i \wedge \overline{dz_j}$ is automatically closed and the expression is meant to describe the integrals over the $\begin{pmatrix} 2n \\ 2 \end{pmatrix}$ different basic homology cycles). Note if $n = 1$ the variety is an elliptic curve, and if $v_1 = 1$ and $v_2 = \tau$ then the determinant evaluates to $\overline{\tau} - \tau$. This is purely imaginary and the real number $h_{1,1}$ can be chosen to scale the magnitude to be a multiple of 2π.

Let M be a compact complex manifold, and suppose there is a positive closed integral $(1,1)$ form. From a previous exercise this comes from a line bundle $\mathcal{L}$ and let X be the dual line bundle $\widehat{\mathcal{L}}$. It contains M as the zero section. The Kodaira embedding theorem (and therefore vanishing theorem) would follow if it could be proved that the identification space where M is contracted to a point is complex analytic.

17. Exercise. Choose a cover of M by open sets V_α and nowhere vanishing sections g_α of $\mathcal{L}$ on V_α. Interpet the g_α as functions on the open subsets U_α which are the inverse image of V_α in the dual line bundle $X = \widehat{\mathcal{L}}$. These define the zero section $M \cap U_\alpha$, so the dg_α/g_α are one forms with simple poles on M of residue 1. Use a partition of unity trick to give smooth (not necessarily holomorphic) $(1,0)$ forms ω_α on U_α with $\omega_\alpha - \omega_\beta = \partial log(g_\alpha/g_\beta)$ on $U\alpha \cap U_\beta$. Show that the $\overline{\partial}\omega_\alpha$ agree on the intersections $U\alpha \cap U_\beta$ to provide a $(1,1)$ form ϕ on X. Can the corresponding quadratic form be used to make a Riemann metric for which points of M are distance zero from each other? Would a Riemannian structure on the singular identification space be way to give it the necessary analytic structure (analagous to how Riemann surfaces can be defined by a metric)?

18. Exercise. Let M be a complex manifold with a $U(1)$ invariant Riemannian metric. Let γ be a real closed smooth curve in M. Choose a holomorphic vector field δ on M such that γ is a real integral curve of δ, parametrized with unit speed. Also suppose there a Kahler potential f defined in a neighbourhood of γ, and let ∂f be its holomorphic differential. Thus locally in coordinates $z_1, ..., z_n$ and time t, $\delta(z_j) = \frac{d}{dt} z_j(\gamma(t))$ while

$$1 = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 f}{\partial z_\alpha \partial \overline{z_\beta}} \delta(z_\alpha) \overline{\delta(z_\beta)}.$$

Let τ be the conjugate vector field on the *complexification* of M, so in local holomorphic coordinates $z_1, ..., z_n$ $\tau(z_i) = 0$ and $\tau(\overline{z}_i) = \overline{\delta(z_i)}$. As γ moves in the complexified manifold according to the flow of τ, show that the rate of change of the holomorphic integral $\int_\gamma \partial f$ is purely real, and is equal to the arc length of γ.

19. Exercise. Show that the $\mathbb{C}$ linear derivations δ on $\mathcal{Y}$ which correspond to actual real vector fields on M are those which commute with σ.

20. Exercise. Show that if δ is such a derivation as in the previous exercise, and if ω is a closed $(1, 1)$ form on M which is invariant under σ (in other words closed and Hermitian), the necessary and sufficient condition for δ to be an isometry for the associated quadratic form on M (which may be a negative or degenerate metric) is that $i_\delta \omega$ is a closed one-form. Here if ω is written $\sum_{\alpha,\beta} h_{\alpha,\beta} dz_\alpha \wedge d\overline{z_\beta}$ as usual, the contraction $i_\delta \omega$ is the one-form $\sum_{\alpha,\beta} h_{\alpha,\beta}(\delta(\overline{z_\beta})dz_\alpha - \delta(z_\alpha)d\overline{z_\beta})$.

21. Exercise. Let M be a complex manifold and let D be any divisor on M. Let ω be the corresponding $\overline{\partial}$-closed $(1,1)$ form and let $C \to M$ be a holomorphic map from a compact Riemann surface. Show that if ω happens to be ∂-closed then $\int_C \omega = 2\pi i m$ where m is the (positive or negative) degree of the line bundle on C which is the pullback of the holomorphic line bundle $\mathcal{O}_M(D)$.

22. Exercise. Many of the formulations in early exercises here did not use that the complex manifold M is nonsingular (this was intentional). How much of Hodge theory still works for singular complex manifolds, such as singular algebraic varieties? (This might be already investigated as it seems an obvious question)