Traces

In this preliminary note, we'll attempt to factorize the Chern character map in a way which is conceptually motivated by the Bass conjecture about what Bass called Hattori-Stallings traces, as such types of traces had been investigated in papers of Hattori and of Stallings.

Let V be a complex manifold, and let Coh be the category of coherent analytic sheaves on V.

For each natural number p, let $\mathcal{D}_p$ be the category of finite length complexes of coherent sheaves X such that the kernel mod image homology $h.(X)$ is locally free[1] with

$$h_i(X) = 0, \text{ for } \ i \neq 0, p$$

while there is an isomorphism

$$\phi : h_p(X) \cong h_0(X) \otimes \Lambda^p \Omega_V .$$

For simplicity let's assume $p \geq 2$, so that any exact sequence in D_p induces an exact sequence on all h_i.

In case $p = 1$ it may be necessary to decree that a sequence is not exact unless it induces an exact sequence on h_0 and h_1.

Let $K_0(\mathcal{D}_p)$ be the free abelian group generated by the objects of $\mathcal{D}_p$ modulo the relation $A - B + C = 0$ whenever there is an exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{D}_p$.

First let's construct a map

$$K_0(Coh) \to K_0(\mathcal{D}_p).$$

The principal parts sheaf of $\mathcal{O}_V$ is the subsheaf of $\mathcal{O}_V \otimes \Omega_V$ spanned by local sections

$$f \oplus df.$$

[1] The requirement for an object of $\mathcal{D}_p$ that the h_i are locally free can be removed throughout without any difficulty. One would define the trace map by using locally free resolutions of objects of $\mathcal{D}_p$

It can also be described as the quotient of $\mathcal{O}_V \otimes_{\mathbb{C}} \mathcal{O}_V$ modulo the square of the ideal sheaf spanned by local sections $(f \otimes 1) - (1 \otimes f)$.

The exterior algebra sheaf $\Lambda^{\cdot}\mathcal{P}(\mathcal{O}_V)$ has both left and right $\mathcal{O}_V$ action. The exact sequence

$$0 \to \Omega_V \to \mathcal{P}(\mathcal{O}_E) \to \mathcal{O}_E \to 0$$

is split on both sides, but not split as a bimodule extension. It induces a differential

$$... \to \Lambda^i\mathcal{P}(\mathcal{O}_V) \to \Lambda^{i-1}\mathcal{P}(\mathcal{O}_V) \to ...$$

and the sequence of augmentation kernels is the $\Lambda^i\Omega_V$.

For any coherent sheaf $\mathcal{F}$, tensoring with $\Lambda^{\cdot}\mathcal{P}(\mathcal{O}_V)$ on one side and using the action on the other side gives an exact but not necessarily split exact sequence with augmentation kernels $\mathcal{F} \otimes \Lambda^i\Omega_V$. This is incidentally an exact and coherent homomorphic image of the truncated Hochschild sequence for $\mathcal{F}$.[2]

The exact sequence $\Lambda^p\mathcal{P}(\mathcal{O}_E) \otimes \mathcal{F} \to ... \to \Lambda^0\mathcal{P}(\mathcal{O}_E) \otimes \mathcal{F}$ gives an object of $\mathcal{D}_p$. Now we can define the map

$$K_0(Coh) \to K_0(\mathcal{D}_p).$$

The map sends a locally free cohrent sheaf $\mathcal{F}$ and a complex number λ to the exact sequence X just described, and the isomorphism

$$\phi : h_0(X) \otimes \Lambda^p\Omega_V \to h_p(X)$$

is the identity.

Secondly, let's define a trace map

$$T : K_0(\mathcal{D}_p) \to H^p(V, \Lambda^p\Omega_V).$$

This will require some verification. We only need to worry about locally free $\mathcal{F}$ and anyway since V is a smooth manifold any $\mathcal{F}$ has a locally free resolution. Each object X of $\mathcal{D}_p$ with $h_0(X) = \mathcal{F}$ has an underlying element $x \in Ext^p(\mathcal{F}, \mathcal{F} \otimes \Lambda^p\Omega_V)$, and the usual

[2] In *Rozansky Witten invariants and the Atiyah* class Kapranov first observes that there is no reason to use an exterior algebra instead of a tensor algebra; here one obtains invariants analagous to the Chern character by using any other Young tableau in place of the one describing the exterior power

theory of Chern characters defines a trace $T(x) \in Ext^p(\mathcal{O}_V, \Lambda^i\Omega_V)$. Thus each object of D_p has a trace already, from the theory of Chern characters. But what needs to be checked is that this trace is additive on exact sequences of elements of $\mathcal{D}_p$ so that the map on objects of $\mathcal{D}_p$ yields a map on $K_0(\mathcal{D}_p)$. Note that an exact sequence of elements of $\mathcal{D}_p$ is a double complex.

A very geometric definition of the action of T on objects is to pull back along an interated projective bundle to split the locally free coherent sheaf $\mathcal{F}$ into a sum of rank one locally free sheaves. The trace of an exact sequence $0 \to \Lambda^p\Omega_V \otimes \mathcal{F} \to \ldots \to \mathcal{F} \to 0$ when $\mathcal{F}$ is locally free of rank one is just the extension class of the sequence twisted by the dual $\widehat{\mathcal{F}}$.

Thirdly, let's define a map $H^p(V, \Lambda^p\Omega_V) \to H^p_{\overline{d}}(V, \Lambda^{p,\cdot}(\mathcal{O}_V \otimes \Omega_{V_\mathbb{R}}))$. Here $V_\mathbb{R}$ is the underlying real manifold, and $H^p_{\overline{d}}$ is the kernel mod image cohomology of the map on global sections induced by the differential

$$\overline{d} : \Lambda^{p,i}(\mathcal{O}_V \otimes \Omega_{V_\mathbb{R}}) \to \Lambda^{p,i+1}(\mathcal{O}_V \otimes \Omega_{V_\mathbb{R}}).$$

It is explained in Griffiths and Harris' book that the $\overline{d}$ Poincare lemma shows that this is an exact sequence of sheaves and it is acyclic. Therefore the kernel mod image homology of the sequence is isomorphic to $H^p(V, \Lambda^p\Omega_V)$. We take the desired map to be the isomorphism in the Griffiths and Harris book, it is an existing isomorphism called the the (p, p)'th Dolbault isomorphism.

Assuming one can verify that the trace map T is well defined one will have a commutative diagram for any complex manifold V

$$\begin{array}{ccc} K_0(\mathcal{D}_p) & \stackrel{T}{\to} & Ext^p(\mathcal{O}_V, \Lambda^p\Omega_V) \\ \uparrow & & \downarrow \\ K_0(Coh) \times \mathbb{C} & \stackrel{Ch}{\to} & H^p_{\overline{d}}(V, \Lambda^{p,\cdot}(\mathcal{O}_V \otimes \Omega_{V_\mathbb{R}})) \end{array}$$

Commutativity can be directly checked as a cocycle calculation; recall that we already mentioned that the exact sequence in the definition of the left vertical map when the coordinate $\lambda \in \mathbb{C}$ is equal to one is the p'th truncation of an exact coherent homomorphic image of the Hochschild complex of the coherent sheaf $\mathcal{F}$.

For any complex manifold V such that the deRham d operator is zero on the $H^p_{\overline{d}}(V, \Lambda^{q,\cdot}(\mathcal{O}_V \otimes \Omega_{V/\mathbb{R}}))$ these vector spaces are subspaces whose direct sum for $p+q=m$ is the Eilenberg-Steenrod cohomology $H^m(V, \mathbb{C})$, in other words the cohomology of the constant sheaf $\mathbb{C}$. Because the complex $\mathcal{O}_V \otimes \Omega_{V/\mathbb{R}}$ with differentials d and $\overline{d}$ resolves $\mathbb{C}$.

The Hodge conjecture asserts that the sum of the terms for $p = q$ is the complex vector space span of the image of the Chern character map. If the map T is really well-defined, this is equivalent to the assertion that $K_0(\mathcal{D}_p)$ is generated by the image of the left vertical map together with objects X whose underlying p-extension of h_0 by h_p has trace zero. Determining which manifolds satisfy this condition requires deciding whether each element of $K_0(\mathcal{D}_p)$ can be written as a sum of elements induced from $K_0(Coh) \times \mathbb{C}$ together with trace zero elements.

It appears possible in some examples to find sequences induced from Coh plus split sequences which do map onto any given element of $\mathcal{D}_p$. In cases when one can find such an object of $\mathcal{D}_p$ mapping onto the kernel, in caseses when it is then possible to continue so that the resolution process finishes in finitely many steps, the desired assertion is true. Therefore, if the trace map T really is well-defined, then the Chern character factorizes into three parts, a K-theoretic part, a trace map, and the Dolbeault isomorphism; and surjectivity of the K-theoretic part would follow an appropriate Devissage argument, if one existed, similar to the argument which already shows that $K_0(Coh)$ is generated by locally free sheaves. I do not yet know whether the desired assertion is in fact equivalent to the existence of a devissage argument. One is allowed to twist the underlying exact sequence of an object of $\mathcal{D}_p$ and also the isomorphism ϕ by any line bundle without affecting the image under T, and for projective varieties this means that any rational map $A - \to B$ in $\mathcal{D}_p$ – meaning one defined over a dense open subset of V – can be converted to an actual map by twisting A by a divisor equivalent to a negative multiple of a hyperplane section to remove the poles. Slightly more abstractly, for each object B of $\mathcal{D}_p$ one can twist any other object A without affecting its image under T, nor affecting the question of whether A is a sum of objects induced from coherent sheaves, such that $\mathcal{H}om(A, B)$ is generated by global sections.

The main point is that the trace map (if it is well-defined) is automatically surjective just because any element of $Ext^p(\mathcal{O}_V, \Lambda^p\Omega_V)$ can be represented by a p extension of coherent sheaves (I believe this was in Verdier's thesis); i.e., when we take $\mathcal{F} = \mathcal{O}_V$ each object essentially equals its own trace.

And the Dolbeault isomorphism is also of course surjective.

The lower map is bilinear and induces a map with source $K_0(Coh)\otimes_{\mathbb{Z}} \mathbb{C}$. It might be helpful to investigate in what possibly weaker sense the leftmost vertical map is also bilinear.

In the case $p = 0$ the vertical map is onto because all 0-extensions are split, and the trace map T is well defined because it is the Hattori-Stallings trace map. The map on K_0 of each local analytic ring is the one which on each connected component of V assigns to each finitely presented module its rank, which is related to the rank of a presenting matrix in the usual way.

If V has connected components $V_1, ..., V_n$ then an element of $H^0(V, \mathbb{C})$ specifies a scalar λ_i for each component V_i. The element is the image under the left vertical map of the sum of the pairs $\mathcal{F}_i, \lambda_i$ where $\mathcal{F}_i$ is taken to be $\mathcal{O}_{V_i}$, supported on the i'th component.

Although the Hodge conjecture is only stated for Kahler manifolds, the K theoretic statement is still equivalent to surjectivity of the Chern character map which is the lower arrow in the commutative diagram, even if V is just an arbitrary complex manifold, as both the trace map and the Dolbeault isomorphism are surjective for any complex manifold. Assuming that the map T is well defined, the corresponding version of the Bass conjecture, if somehow proven true, would extend the Hodge conjecture to arbitrary complex manifolds and provide a proof of the extended statement.

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