

Comments about Hilbert's 16'th problem

*John Atwell Moody*  
*Warwick Mathematics Institute*  
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## 1. Introduction.

Hilbert's 16'th problem has two familiar parts

“Was die Curven 6ter Ordnung angeht so habe ich mich – freilich auf einem recht umständlichen Wege...”

and

“Im Anschluss an dieser rien algebraische Problem...”

The second part is a hard and unanswered question of real analysis; it will require a subsequent survey article to this one to begin to unravel some of what has been done.

The precursor was work of Darboux [1] twenty-two years earlier, and Poincare [2], who looked for singular complete curves of degree  $e$  invariant under a vector field with poles of degree  $d$ , saying that the problem ‘n’a pas attire l’attention des geometers autant quelle meritait,’ but ‘serait resolu si l’on avait, dans tous les cases, une limite superior du nombre  $e$ .’

Cerveau and Lins Neto [17] and Carnicer [16] solved Poincare’s problem for nodal curves, and for generic vector fields. If the curve has nodal singularities, or if there are finitely many local invariant curves through each singular point of the foliation,  $e$  is never more than<sup>1</sup>  $d + 3$ . Both sets of authors, I think, knew that  $e$  would not be bounded in terms of  $d$  in general. Lins Neto asked, then [9], are there are vector fields with fixed  $d = 3, 4, \dots$  and arbitrarily large  $e$  with no rational first integral.

Lins Neto’s question was answered by Ollagnier [10], who had already found necessary and sufficient conditions for the special type of quadratic vector field which are the Lotka-Volterra equations from biology, to have no rational first integral, and these include cases when there are invariant curves of arbitrarily high degree. (An independent solution was is in [14]).

Therefore, even if we restrict the second part of Hilbert’s question to algebraic limit cycles, it is still not possible to argue that the absence of any rational first integral for limit cycles implies them few in number by reasons of degree.

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<sup>1</sup>We are using ‘degree’ to mean the degree of the divisor of poles of the vector field, which is one less than the number in Carnicer’s paper

The insightful analysis [11] of Christopher, Llibre and Swirszcz shows that the degree of a curve by itself is not a meaningful invariant after all. The quadratic transform does not change the dynamical behaviour of a vector field, while it does change the degree of invariant curves. Also the paper includes an interesting argument over the reals for the existence of an algebraic curve of degree 12 which is invariant of a particular quadratic vector field with no rational first integral, but such that there is no second invariant algebraic curve.

In this note, we'll restrict to the case of complete invariant holomorphic curves (not necessarily normal or irreducible though). The invariant curve has a resolution in which it remains invariant and the vector field does not acquire any new poles, although the foliation does not become nonsingular.

There is a condition necessary and sufficient for the existence of a rational first integral, in terms of one forms on  $\mathbb{P}^2$  twisted by  $2eH$  for  $H$  a line; yet it seems difficult to approach this without removing some of the twisting.

Less extreme log twisting, once analytic solution germs  $\gamma_i$  are given<sup>2</sup> at singular points  $p_1, \dots, p_s$  of the foliation (here  $s \leq d^2 + 3d + 3$ ), shows how to find any complete singular invariant holomorphic curve of degree<sup>3</sup> larger than  $\frac{1}{2}\sqrt{9 + 4S} + d + \frac{3}{2}$  passing through only them, by rational integration, where  $n_i$  is the maximum by which the local degree of the germ exceeds the (positive) order of  $\delta$  at points infinitesimally close to  $p_i$  and<sup>4</sup>  $S = \sum_{i=1}^s (n_i - 1)(n_i - 2)$ .

In a real polynomial dynamical system of degree  $d$  with a limit cycle, if there were ever a way of learning the maximum discrepancy  $n_1, \dots, n_s$  over the set of algebraic germs, products of at most  $\binom{d+2}{2} + 1$  parts by Darboux theory, one could put the largest  $n_1, \dots, n_s$  in the formula  $S$ . As soon as any more than  $\frac{1}{2}d\sqrt{9 + 4S} + \frac{1}{2}d^2 + \frac{1}{2}S + 2$  limit cycles occur, as many as the remainder could never be continued to complete complex curves, by Harnack's theorem.<sup>5</sup>

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<sup>2</sup>It may be confusing to say they are 'given' because then there is at most one solution. I mean, the Zariski closure of the  $\gamma_i$  is either  $\mathbb{P}^2$  or a lower degree curve, or else the  $\gamma_i$  can be simultaneously  $\delta$  equivariantly deformed.

<sup>3</sup>If all  $n_i < 3$  this simplifies to  $d + 3$ .

<sup>4</sup>Terms where  $n_i \leq 0$  should be removed from the sum.

<sup>5</sup>Seven terms cancel.

## 2. Comment on the history of the problem.

The history is maybe more important than the question itself. Changes in our interpretation of ‘integrable’ and ‘rational’ show the old question in relief. From the first part of Hilbert’s question, about real algebraic curves of degree six, to second, seems now as if stepping into a meaningless abyss, without any reason to have done so. In maths of the Greek empire, people thought that theorems had political content, and in Hilbert’s questions one sees now blind hope for some sort of guidance for what we should have done, or what will happen next.

A point in the plane, under the influence of a polynomial vector field, Hilbert must have wondered, should not be just rattling aimlessly, in the way we now understand the more modern Lorenz butterfly effect.<sup>6</sup>

The more modern theory of dynamical systems makes a distinction too, though. It is not any more seeking to find guidance, or analyzing when it may be found or lost, but rather accepting and doctrinizing that there is a, possibly natural, system, and within it, points acting as particles, helplessly obeying it anyway.<sup>7</sup>

Now the distinction between separate models is that they each comprise different hypotheses, differing from one another as one changes one’s mind, without any continuity; and presenting diverse and discrete choices for human intervention into nature, which we may continue to advertise to each other in various ways.

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<sup>6</sup>The Poincare-Bendixson theorem may have been known by then although it was proved one year later.

<sup>7</sup>The Lotka-Volterra model, within which Ollagnier found his counterexample to the question of Lins Neto, is this: there actually is, in the real world, an integer vector made up of the numbers of organisms of each species in existence. We may perform a linear regression to see how the logarithmic derivative with respect to time may depend on the real value of this vector. The Lotka-Volterra assumption is that this can be done without any error, which means that the probability an individual in one species dying, minus the probability of having a child, is a utility function determined by linear regression on the numbers of individuals of all species.

### 3. Chern classes.

If  $\delta$  is a meromorphic vector field on the projective plane with divisor<sup>8</sup>  $K_1$  of degree  $d$ , the singular subscheme of the underlying singular foliation is the algebraic cycle

$$c_2 - (K_1) \cdot (c_1 - K_1)$$

where  $c_1, c_2$  are first and second Chern classes of the projective plane. The three coordinate lines make a triangle of projective lines, and we can find a one dimensional flow fixing the corners, or a two dimensional flow preserving the lines themselves. So we may take as  $c_2$  the union of the corners and as  $c_1$  the union of the lines. The calculation above gives the answer of  $c_2$  when  $K_1$  is empty and adjusts it as  $K_1$  enters the picture.

If we take  $e$  to be a natural number, the necessary and sufficient condition for there to exist a divisor linearly equivalent to  $eH$  with a component belonging to a 'level set' of a rational function, is the condition that there are two non-linearly-dependent  $f, g \in \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(eH))$  such that

$$f\delta(g) = g\delta(f).$$

To make best sense of this we explicitly interpret global sections of  $\mathcal{O}_{\mathbb{P}^2}(eH)$  as rational functions with poles no worse than  $eH$ .

Therefore there is a value of  $e$  for which this holds if and only if there is a rational first integral for  $\delta$ .

The sheaf of one forms on the projective plane with poles no worse than  $2eH$  has a vector space of global sections of dimension  $4e^2 - 1$ , and although  $fdg$  and  $gdf$  may have poles of order higher than  $2eH$ , the difference  $fdg - gdf$  always has poles no worse than  $2eH$ . Therefore, we can translate the necessary and sufficient condition for rational integrability a little bit: we can find within the projectivication of

$$\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2eH) \otimes \Omega_{\mathbb{P}^2}^1)$$

a copy of the Grassmannian variety of two planes in  $\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(eH))$ . The two-plane spanned by  $f, g$  is sent to the line spanned by  $fdg - gdf$ .

There is a map of line bundles inducing a map of locally free coherent sheaves

$$\Omega_{\mathbb{P}^2} \rightarrow \mathcal{O}(K_1)$$

with kernel isomorphic to  $\mathcal{O}_{\mathbb{P}^2}(-3H - K_1)$ , and whose cokernel is the actual coordinate ring of the algebraic cycle described above (twisted by  $K_1$  which has a trivial effect).

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<sup>8</sup>Here  $d$  is the order of poles minus the order of zeroes – yet the 'degree' of the vector field by the usual terminology is  $d + 1$  not  $d$ .

The way that the calculation of the singular locus of the foliation is related to the necessary and sufficient condition for rational integrability is that we twist the exact sequence here by  $\mathcal{O}_{\mathbb{P}^2}(2eH)$ , pass to global sections, and we obtain then by restriction to the Grassmannian subvariety, a rational map from a Grassmannian variety to projective space; and the equivalent condition of existence of a rational first integral is that this rational map *fails* to be a morphism.

When there is no rational first integral; i.e. when the map from the Grassmannian is a morphism, the fibers are codimension zero or dimension zero since the Picard group of the Grassmannian is cyclic with (very) ample generator.<sup>9</sup>

There is nothing interesting to observe from the standpoint of dimension only; the Grassmannian has dimension  $e^2 + 3e - 2$ , the projectivized one forms have dimension  $4e^2 - 2$ , and the rational map goes to  $\mathbb{P}\Gamma\mathcal{O}_{\mathbb{P}^2}(2eH + K_1)$  of dimension  $\binom{2e + d + 2}{2} - 1$ . Although the projectivized one forms twisted in this manner do not acquire enough global sections to include the Plucker embedding, the projective embedding associated to the very ample generator of the Picard group, what happens is that the Grassmannian is still faithfully represented in the  $4e^2 - 2$  dimensional projective space after a linear subspace is projected away, perhaps after a projective normalization.

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<sup>9</sup>This observation was made by D. Maglagan in a seminar, and the subsequent discussion is based on a further comment by M. Reid; in fact the comment mentioned Whitney's trick which I now suspect could relate the indeterminacy to higher Chern classes.

#### 4. The case of logarithmic twisting

Let us see what can be done if we take global sections without so much twisting, but only a bit of log twisting. When we do this, some information will be lost, but not all. In order to calculate things, we resolve our curve let us call it  $C_1$  and so replace  $\mathbb{P}^2$  by the surface  $S \rightarrow \mathbb{P}^2$  obtained by blowing up singular points of  $C_1$  or the total transform of  $C_1$  which are more than simple crossings. This does not increase the poles of  $\delta$ , in fact since the chosen points are invariant the flow continues to preserve the partial resolution.

The total transform of  $C_1$  is a simple normal crossing divisor which we again call  $C_1$ . It has coefficients which are positive numbers; each prime divisor  $P$  in the total transform of  $C_1$  is given the coefficient which is the valuation at  $P$  of any local defining equation of  $C_1$ . We will abbreviate this  $\nu_P(C_1)$ .

The locally free sheaf of logarithmic one forms  $\Omega_S(\log C_1)$  where here  $C_1$  abbreviates the pullback or total transform of  $C_1$ , has a map to  $\mathcal{O}(K)$  where  $K$  the divisor of  $\delta$  on  $S$  is  $K_1 - Z$  where  $Z$  is the divisor of extra zeroes which  $\delta$  acquires on  $S$ .

The first Chern class of  $\Omega_S(\log C_1)$  is the reduced divisor underlying the total transform of  $C_1$  plus the canonical divisor of the projective plane  $-3H$  plus the ramification which is the exceptional part of the reduced divisor underlying the total transform. That is, if we denote the reduced exceptional divisor as  $E$  the first chern class is

$$-3H + E + (C + E) = -3H + E + (eH - I) + E.$$

The first Chern class of the other sheaf is  $K = K_1 - Z$ , so the difference, taking  $K_1 = dH$ , is

$$(e - d - 3)H + (2E + Z - I).$$

The argument of Bogomolov [6] theorem 12.2 which works in any dimension is also explained in Miyaoka's paper in a simpler way for surfaces [8] theorem 2, page 230 attributed to Reid's [7] Proposition 2, written shortly after his thesis, and attributed to a discussion<sup>10</sup> by Barth, Peters and Van de Ven [12] of Castelnuovo [4] p. 501. If  $w_1, w_2$  are closed forms in the same line bundle then there is a rational function  $g$  with  $w_2 = gw_1$  then

$$0 = dw_2 = gdw_1 + dg \wedge w_1 = dg \wedge w_1$$

This means that  $dg$  is a rational section of the kernel hence

$$0 = \delta(g).$$

Although Castelnuovo wrote just five years after Hilbert did, he likely knew already that Hilbert was asking two unrelated questions. However, if he had not known that – if he had wanted to apply his work, or if he had been motivated by trying to unite the two parts – one of the missing ingredients would be the degeneration of the Hodge deRham spectral sequence [5] 3.12 (ii), what is, I think part of Deligne's thesis.

Deligne even observes the specific corollary needed in “cas particulier  $E_1^{p0} = E_\infty^{p0}$ ” in Coro 3.2.14, page 39: logarithmic forms are always closed.

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<sup>10</sup>This refers to the earlier edition of Barth Peters and Van de Ven. The current edition of Barth Hulek and Van de Ven includes now the full text of [7] verbatim.



To summarize,

**5. Theorem.** For  $\delta$  any plane vector field of degree  $d$ , for  $p_1, \dots, p_s$  singular points of the foliation, and  $\gamma_1, \dots, \gamma_s$  analytic solution germs at  $p_1, \dots, p_s$ , suppose that there is a nontrivial pencil of plane curves  $C_2$  of degree  $q$  whose order at each prime  $P$  in the resolution of  $p_i$  is at least

$$\nu_P(\gamma_i) - \nu_P(\delta) - 2,$$

then any complete solution curve of degree  $q + d + 3$  passing through only the  $\gamma_i$  can be found by rational integration.

**6. Corollary** For  $\delta$  a plane vector field of degree  $d$ , and  $\gamma_1, \dots, \gamma_s$  analytic solution germs at  $p_1, \dots, p_s$ , let  $n_i$  be the maximum by which the order of  $\gamma_i$  at points  $P$  infinitesimally close to  $p_i$  exceeds the (positive) order of  $\delta$ . Any solution curve of degree larger <sup>11</sup> <sup>12</sup> than

$$\frac{1}{2} \sqrt{9 + 4 \sum_{i=1}^s (n_i - 1)(n_i - 2) + d} + \frac{3}{2}$$

passing through only the  $\gamma_i$  can be found by rational integration.

*Proof.* For each number  $e$  the complete linear system of degree  $e - d - 3$  plane curves has dimension  $\binom{e - d - 1}{2} - 1$ . The pencil of curves in the theorem includes the pencil of ordinary plane curves through the base cycle  $(n_1 - 2)p_1 + \dots + (n_s - 2)p_s$ . The dimension of this is unknown, depending on the special position of the  $p_1$ , but the number is of classical interest. Subtracting the sum of the  $\binom{n_i - 1}{2}$  as if the adjunction conditions were linearly independent, the familiar quadratic formula gives the value of  $e$  when the linear system always begins to move even if the  $p_i$  are in general position.

**7. Remark.** The global one forms on  $\mathbb{P}^2$  with logarithmic poles on  $C_1$  are included in the global one forms on  $S$  with logarithmic poles on  $C + E$ . The local monomialization  $f = \prod x_i^{e_i}$  expresses  $\frac{df}{f}$  as a linear combination of the  $\frac{dx_i}{x_i}$  and these patch together. Therefore any two-plane of global logarithmic forms in the kernel of  $\Omega_{\mathbb{P}^2}(\log C_1) \rightarrow \mathcal{O}(K_1)$  arises geometrically in the manner of the theorem, and all such logarithmic one forms are closed, even though  $\Omega_{\mathbb{P}^2}(\log C_1)$  may not be a locally free sheaf.

<sup>11</sup>Terms where  $n_i \leq 0$  should be removed from the sum.

<sup>12</sup>As we remark in another footnote, when all  $n_i < 3$  this simplifies to  $d + 3$ .

**8. Remark.** The map  $\Omega(\log C_1) \rightarrow \mathcal{O}(K_1)$  exists even though  $C_1$  does not have normal crossings, nor is  $\Omega(\log C_1)$  locally free. If  $F$  is principal supported on  $C_1$  and  $f$  the unique nonzero global section of  $\mathcal{O}(F)$  up to scalars then,  $\frac{df}{f}$ , maps to a global section of  $\mathcal{O}(K_1)$ . Darboux observed, if things are defined over  $\mathbb{Q}$ , that once the number of components of  $C_1$  is more than one (the rank of the Picard) group plus the dimension of  $\Gamma(\mathcal{O}(K_1))$ , there are global log forms in the kernel. Also a completely explicit and more general theorem is in [13]. The kernel is contained in the kernel of the corresponding map on the resolution and therefore Darboux' theory can explicitly construct the pencil  $C_2$  and in turn the indeterminacy on the Grassmannian in cases when  $C_1$  has many components.

**9. Remark.** It is probably not true that arguments about global sections of  $\Omega(\log C + E)$  can give the precise necessary and sufficient conditions for a rational first integral in the manner in which we know that arguments about  $\mathcal{O}(2C_1) \otimes \Omega$  can. The technique of resolutions is just a way of short-cutting the more difficult Grothendieck group calculation, but the sensible approach may be to bring in duality and bilinear algebra to try to express the indeterminacy of the map on the Grassmannian in terms of local contributions where the vector field meets the curve.

## 10. Conclusion.

In writing this, I understood a bit about something that the biologist Jack Cohen once told me. He arrived under the protection of Ian Stewart, and Jack told me that he had resigned his position at another university. His job there, he said, included assigning a degree classification, a percentage, to each student. In a meeting, it had been noticed that the distribution of grades which he assigns does not resemble a Gaussian distribution, or even a smooth function. He had been asked to assign numbers a different way.

My worry, along the lines of the Lotka-Volterra assumption, had been this: even if we do away with the linearity assumption, there is still the issue of *Frobenius* integrability which has to be applied to the construction of the utility function of each organism, or each person. In the failure of *Frobenius* integrability it does not even make sense to speak of a utility *function* at all. Jack did not say anything, but he brought me a petri dish, containing little creatures, hydra, daphnia, and many little creatures and plants. They were interacting, and almost playing with each other.

I knew this already, as an American, having come from a country which was nearly a natural wilderness when I was a child, and I had seen even more the overwhelming combination of complexity and meaning which is in nature. The indefinite boundary between what we thought to be land or sea, the creatures, the marshes and rivers, the plants, the tides, the sunlight, all changing together in meaningful ways that could not be described in writing.

Jack said, he has been paid, and taken money, to help create duck ponds, where a stream passes through. He said, there is usually a sort of machine, so that the water has to go underground to leave the lake, and owners hate this, they remove it. But then the trough beneath begins to fill with silt, and they call on Jack as a consultant. He said, he takes the payment and advises them to reinstate the machinery. He said, there is such a duck pond here; it was the pride of our university. Since then a small area has also returned to nature.

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