

Finite generation and the Gauss process ¹

Abstract: Convergence of the Gauss resolution process for a complex singular foliation of dimension r is shown to be equivalent to finite type of a graded sheaf which is built using base $(r + 2)$ expansions of integers. As applications it is calculated which foliations coming from split semisimple representations of commutative Lie algebras can be resolved torically with respect to an eigenspace decomposition and it is shown that Gaussian resolutions stabilize for irreducible projective varieties with foliations of dimension r for which $(r + 1)H + K$ has Iitaka dimension less than two where H is a hyperplane section and K a canonical divisor of the foliation. Another application is that for normal irreducible complex projective varieties with very ample divisor H and a resolvable foliation, there are functorial locally closed conditions on vector subspaces $X \subset |iH|$ which hold for large i and ensure that blowing up the base locus of X and one further Gaussian blowup resolves the foliation.

1. Introduction.

Let V be an irreducible algebraic variety over a field k of characteristic zero, furnished with a singular foliation.

The *Gauss blowup* of the foliated variety V is the image of the Gauss rational map to a Grassmannian, or rather the lowest domain of definition of that map. In reasonable cases a foliation which can be resolved can be resolved by its Gauss map, but this is not always so. A necessary and sufficient condition for resolvability of a singular foliation by an arbitrary locally projective birational morphism in characteristic zero was described in an earlier paper [1], that there is an ideal sheaf I and a number N so that

$$I^N \mathcal{J}(I)^{r+2} = I^N \mathcal{J}(I\mathcal{J}(I)).$$

¹John Atwell Moody, Maths, Warwick University; moody@maths.warwick.ac.uk

The motivating goal of this paper is to show that if the condition holds for some value of N then there is some possibly different choice of I so that it holds with $N = 0$.

Working, initially from affine examples, we shall construct a sheaf $R(I)$ of graded algebras, depending on I , making use of base $(r + 2)$ expansions of integers, such that finite generation ('finite type') of $R(I)$ over k is equivalent to convergence of the resolution process where first one blows up V along I and then follows by iterating the Gauss process. When this is the case, the locally projective morphism associated to the graded sheaf turns out not to be the limit (=last stage) of the tower, but rather it is a variety which converts the next-to-last stage to the last stage by pulling back.

It follows from the construction that resolving the foliation by a locally projective morphism is equivalent to solving the simpler equation $\mathcal{J}(I)^{r+2} = \mathcal{J}(I\mathcal{J}(I))$. The simpler condition is now neither necessary nor sufficient for I itself to be a resolving ideal.

The operator \mathcal{J} in [1], which operated upon fractional ideals, depended upon a basis of derivations of the rational function field of V over k . We will replace it with a more functorial version of the same operator: therefore with a functor F .

In this introduction, which will be light reading, we won't define the functor F ; it is enough to know that it operates on torsion free rank one coherent sheaves on V . Then the graded sheaf R also depends functorially on a torsion free coherent sheaf I , and let us describe how things fit together.

Each graded part R_i is also to be functor operating on torsion free rank one coherent sheaves, such that the i 'th term of $R(I)$ is the functor R_i applied to the sheaf I . Here is how the functor R_i is defined, starting with the functor F . If H and G are functors acting on torsion free coherent sheaves on V of rank one, we will always define the product HG to be the most obvious functor which assigns to each J the sheaf $H(J)G(J)$ by which we mean the tensor product of $H(J)$ and $G(J)$ reduced modulo torsion. Note that the identity functor id is not an identity element for the multiplication.

In terms of this product operation, define a sequence of functors L_i

by the inductive rules

$$\begin{aligned} L_0 &= F \\ L_{i+1} &= F(\text{id } L_0 \dots L_i) \end{aligned}$$

Then for each number i define the functor

$$R_i = L_0^{a_0} L_1^{a_1} \dots L_s^{a_s}$$

where $i = a_0 + a_1(r+2) + \dots + a_s(r+2)^s$ with $0 \leq a_\alpha < (r+2)$ is the base $r+2$ expansion of the degree i . For each choice of sheaf I we show the $R_i(I)$ fit together to be a graded sheaf of algebras.

The multiplication law is to be based on the carrying operation which takes place when two finite geometric series to the base of $(r+2)$ are added. The carrying operation in the s 'th place, for a torsion free coherent sheaf J , will require a map

$$F(JL_0(J) \dots L_s(J))^{r+2} \rightarrow F(JL_0(J) \dots L_{s+1}(J))$$

satisfying suitable compatibility conditions for the various values of s coming from the ring axioms. The fact that the left and right sides of the map above have the same degree in the graded sheaf R requires that the action of F on degrees will be the affine transformation of multiplying by $r+1$ and then adding 1, corresponding to the numerical identity

$$\begin{aligned} &(1 + (r+1)(1 + (r+2) + \dots + (r+2)^s))(r+2) \\ &= 1 + (r+1)(1 + (r+2) + \dots + (r+2)^{s+1}). \end{aligned}$$

A goal of later sections will be to begin to understand a manner in which the graded sheaf $R(I)$ and the numerical identity, may have a meaningful representation together in the context of algebraic geometry.

The more precise statement of our result, now, is that starting with a singularly foliated irreducible variety V over a field k of characteristic zero, furnished with a torsion free rank one coherent sheaf I , the graded sheaf $R(I)$ is finite type over k if and only if that sequence of blowups of V stabilizes, which consists of blowing up

I and then taking successive Gaussian blowups with respect to the foliation.

As a test application of the picture, we will construct the sheaf $R(\mathcal{O}_V)$ when V is the cusp, and also in the case of the simplest non-resolvable foliation of the plane and other examples. Then we will apply the rule to determine precisely which of the foliations coming from split semisimple representations of commutative Lie algebras can be resolved torically, with respect to the toric structure coming from an eigenspace decomposition. The answer will show us that those which can be resolved torically can all be resolved by one single Gauss blowup.

Next, taking V to be a normal irreducible quasiprojective variety, and I to be a very ample line bundle, we will show that $R(I)$ is always generated by global sections. Therefore the finite type property for $R(I)$, and in turn also convergence of the Gauss process itself, is equivalent to finite generation of the algebra of global sections.

Next assume the quasiprojective variety V is actually projective and set $D = (r + 1)H + K_V$ where H is a hyperplane section and K_V is the canonical divisor of the foliated variety (in the sense of the introductory pages Bogomolov & McQuillan [16], for example).

It is possible to practically dispense with the cases when the Iitaka dimension of D is zero or 1.

If the Iitaka dimension of D is equal to zero, then the graded algebra in question is an integral domain of transcendency one degree over k ; this can be nothing but a polynomial algebra $k[T]$ in one variable T if k is algebraically closed. This is finite type over k so the Gauss process for V must stabilize; but analyzing further we see that in this case the foliation is always nonsingular.

This is because the letter T can be seen as a global section of $\mathcal{O}_V(K_V + (r + 1)H)$, and the fact that the degree one monomials of the polynomial algebra are spanned by T means this global section spans a complete linear system which generates the sheaf of r 'th exterior power of the differentials of V along the foliation mod torsion. The sheaf is therefore trivial of rank one, and the foliation nonsingular.

If we assume in addition that the foliation is codimension zero, ie

that $r = \dim(V)$, this would imply that the Nash blowups of V itself stabilize, but this is for a trivial reason, for then V can be nothing but projective space (see [18] chapter 3). In our situation this is easily seen as follows, as Kobayashi and Ochiai [6] applies to the nonsingular variety V . They show how to use Hirzebruch Riemann Roch, Serre Duality and Kodaira vanishing, to calculate the Euler characteristic $\chi(\mathcal{O}_V(iH))$. Since it is a polynomial of degree r in i which takes the value 0 when $i = -1, -2, \dots, -r$ and 1 when $i = 0$ then

$$\chi(\mathcal{O}_V(iH)) = \binom{r+i}{i}$$

The multiplicity is equal to the highest degree coefficient so they may conclude

$$H^r/r! = 1/r!$$

showing $H^r = 1$. This shows that V meets an intersection of r hyperplanes at a single point, so is a linear projective space.

For cases when D has nonzero Iitaka dimension, one needs to relate the sheaf of graded algebras $R(\mathcal{O}_V)$ to the graded ‘pluricanonical sheaf’ $\oplus_i \mathcal{O}_V(iK_V)$.

Let then V be a normal irreducible projective variety over a field k of characteristic zero, with a singular foliation, and H a very ample divisor. Let R be the graded sheaf corresponding to the structure sheaf \mathcal{O}_V . Each R_i can be represented as a subsheaf of $\mathcal{O}_V(iK_V)$, compatibly with multiplication of sections, so R can be represented as a subsheaf of graded rings of $\oplus_i \mathcal{O}_V(iK_V)$. Correspondingly the sheaf of rings $R(\mathcal{O}_V(H))$ will be a subsheaf of $\oplus_i \mathcal{O}_V(iD)$ for

$$D = K_V + (r+1)H$$

and we will find that the subsheaf is generated, as a graded coherent sheaf, by global sections.

In this way we obtain a subalgebra of

$$\oplus_{i=0}^{\infty} H^0(V, \mathcal{O}_V(iD))$$

for this divisor D whose finite generation controls stability of the Gaussian resolution process for the foliated variety V .

We may take the subalgebra to be generated by particular global

sections whose degrees are powers of $(r+2)$, defined explicitly, inside the full algebra of global sections of $R(\mathcal{O}_V(H))$.

Here is a geometric interpretation. Abbreviate $L_i = L_i(\mathcal{O}_V(H))$, and $R_i = R_i(\mathcal{O}_V(H))$ omitting the sheaf $\mathcal{O}_V(H)$ by abuse of notation. Since $L_i = R_{(r+2)^i}$ each L_i is generated by global sections.

The product

$$\begin{aligned} & \Gamma(L_0)\dots\Gamma(L_s) \\ & \subset \Gamma(\mathcal{O}_V((1 + (r+2) + (r+2)^2 \dots + (r+2)^s)D)) \end{aligned}$$

is an incomplete linear system. Let W_s be the rational image of V in the corresponding projective space. The supremum of the dimensions of the W_s is at most the Iitaka dimension of D .

The rational map $V \rightarrow W_s$ lifts to a lowest morphism $V_{s+1} \rightarrow W_s$, and the tower of Nash blowups $\dots V_2 \rightarrow V_1 \rightarrow V_0 = V$ is induced from the tower of projective (but not birational) morphisms $\dots W_2 \rightarrow W_1 \rightarrow W_0 \rightarrow \text{point}$.

Stabilization of the tower of V_s is equivalent to finite type for the graded sheaf $R = R(\mathcal{O}_V(H))$ while stabilization of the tower W_s is equivalent to finite type for the graded algebra of global sections. It is a triviality (from both the algebraic and geometric point of view) that stability for the W_s is equivalent to stability for the V_s .

When D has Iitaka dimension equal to one, the tower $\dots V_2 \rightarrow V_1 \rightarrow V_0 \rightarrow V$ must stabilize because $V_{s+1} \rightarrow V_s$ is finite for $s \geq 1$.

I should also remark, the result has no reasonable application to the case of Nash blowups, i.e., to the case when $r = \text{dimension}(V)$, because [18] 8.5.5 shows that $D - H$ is basepoint free, so D is very ample, and the Iitaka dimension of D is r itself, for any normal projective variety V which is Gorenstein with at worst isolated irrational singularities.

2. Definition of F

In this section we will define the functor F . Let V be an irreducible variety over a field k . Choose a K linear sub Lie algebra (actually any subspace will do if one is willing to consider distributions rather than foliations) $\mathcal{L} \subset \mathcal{D}er_k(K, K)$ where K is the function field of V . This corresponds to a singular foliation on V . This and other basic definitions can be found in [1]. Let

$$r = \text{dimension}_K(\mathcal{L}).$$

For coherent sheaves A, B on V we define the product

$$AB \stackrel{\text{def}}{=} (A \otimes B)/\text{torsion}.$$

Let Ω be sheaf of differentials along the foliation, ie the image of the natural evaluation map to the dual vector space $\widehat{\mathcal{L}}$.

$$ev : \Omega_{V/k} \rightarrow \widehat{\mathcal{L}}$$

$$dx \mapsto (\delta \mapsto \delta(x)).$$

for x a local section of the struture sheaf of V .

For any torsion free rank one coherent sheaf I let $P(I)$ be the sheaf of first principal parts of I with respect to the foliation. Thus $P(I)$ is the middle term of the sequence

$$0 \mapsto I\Omega \rightarrow P(I) \rightarrow I \rightarrow 0$$

defined by $\alpha = ev_*(\beta)$ where $\beta \in Ext^1(I, I \otimes \Omega_{V/k})$ is the Atiyah class of I . Now let

$$F(I) = \Lambda^{r+1}P(I)/\text{torsion}.$$

Note F defines a functor which acts on the full subcategory of torsion free rank one coherent sheaves on V .

In [1] we defined a less natural operator called \mathcal{J} acting on fractional ideals (ie nonzero finitely-generated submodules of the R module

K), which depended on a generating basis $\delta_1, \dots, \delta_r \in \mathcal{D}er_k(K, K)$ of the foliation. If V is affine and I is an ideal in the coordinate ring, the fractional ideal $\mathcal{J}(I)$ is the one generated by determinants of matrices whose rows are $(f, \delta_1 f, \dots, \delta_r f)$ for $f \in I$. The span of such rows themselves is a copy of $P(I)$ and the morphism $P(I) \rightarrow I$ in the exact sequence above sends this row to its first entry.

3. Definition of R

Now that we have defined multiplication of coherent sheaves and the functor F , define torsion free rank one coherent sheaves J_i and L_i for $i = 0, 1, 2, 3, \dots$ by the inductive rules

$$\begin{cases} J_0 = I \\ J_{i+1} = IL_0L_1\dots L_i \\ L_i = F(J_i) \end{cases} \quad (1)$$

These depend functorially on I and like $L_0 = F$ can be viewed as functors acting on rank one coherent sheaves. Also for $i = 0, 1, 2, \dots$ define the rank one torsion free coherent sheaf R_i by the rule

$$\begin{cases} R_0 = \mathcal{O}_V \\ R_i = L_0^{a_0}L_1^{a_1}\dots L_s^{a_s} \end{cases} \quad (2)$$

where the numbers a_α are chosen so that the base $(r+2)$ expansion of i is

$$i = a_0 + a_1(r+2) + \dots + a_s(r+2)^s$$

with $0 \leq a_\alpha < (r+2)$.

1. Lemma If the characteristic of k is zero there is a multiplication map for each i, j

$$R_i R_j \rightarrow R_{i+j}$$

which makes

$$R = R_0 \oplus R_1 \oplus \dots$$

into a graded sheaf of \mathcal{O}_V algebras.

Proof of Lemma 1. First suppose V is affine, with affine coordinates x_0, \dots, x_n . To make our formulas work assume x_0 is the constant function $x_0 = 1$. Also choose a K basis $\delta_1, \dots, \delta_r \in \mathcal{L}$. When $I \subset K$ is the fractional ideal generated by a sequence of rational functions y_0, \dots, y_m the isomorphism between $P(I)$ and the span of the rows $(f, \delta_1 f, \dots, \delta_r f)$ induces by passage to highest exterior powers an embedding

$$F(I) \subset K$$

with image the fractional ideal $\mathcal{J}(I)$ generated by determinants

$$h = \text{determinant} \begin{pmatrix} f_0 & \delta_1 f_0 & \dots & \delta_r f_0 \\ \dots & \dots & \dots & \dots \\ f_r & \delta_1 f_r & \dots & \delta_r f_r \end{pmatrix} \quad (3)$$

where f_0, \dots, f_r run over all pairwise products $x_i y_j$. An observant reader would object that the definition of $\mathcal{J}(I)$ on the previous page should require the f_i to run over all the elements of I , however because of [1] Proposition 11, and because $x_0 = 1$ the determinants displayed above generate the whole of $\mathcal{J}(I)$ when the f_i run over the smaller set of pairwise products.

The following calculation of h^{r+2} can be deduced from (3)

$$h^{r+2} = \text{determinant} \begin{pmatrix} h f_0 & \delta_1(h f_0) & \dots & \delta_r(h f_0) \\ \dots & \dots & \dots & \dots \\ h f_r & \delta_1(h f_r) & \dots & \delta_r(h f_r) \end{pmatrix}.$$

It is proved as follows: by multilinearity of the determinant we can argue as if the δ_i commute with h , the commutators of the δ_i and h cancel out, and so the right side of this equation is h^{r+1} times the right side of (3). The left side of (3) is just h . So the determinant evaluates to the product $h^{r+1} \cdot h = h^{r+2}$. This argument is central to both [1] and the current paper.

The ‘polarization identity’ in linear algebra then expresses any homogeneous polynomial of degree $r + 2$ over the rational numbers as a linear combination of such powers h^{r+2} where h is a linear form. This proves any homogeneous polynomial of degree $r + 2$ in $\mathcal{J}(I)$ belongs to $\mathcal{J}(I\mathcal{J}(I))$. In this way we obtain an a priori non-natural map

$$F(I)^{r+2} \cong \mathcal{J}(I)^{r+2} \rightarrow \mathcal{J}(I\mathcal{J}(I)) \cong F(IF(I)).$$

It remains now to verify naturality.

Let's now show the definition is independent of choice of the functions x_i and the rational functions f_j . A simpler but infinite formula not using either choice would have given the same answer as (3) if one allows the f_i on the right side of the equation to run over all elements of the fractional ideal (y_0, \dots, y_m) and not only the products $x_i y_j$ (see again [1] Proposition 11). This shows the choice of x_i and f_j is inessential.

We now need to show the map is independent of choice of basis of \mathcal{L} .
 If τ_1, \dots, τ_r were a different basis then

$$\tau_i = \sum_j a_{ij} \delta_j$$

with $a_{ij} \in K$. Then the images of $F(I)^{r+2}$ and $F(IF(I))$ in K are both multiplied by the same rational function

$$\det(a_{ij})^{r^2+3r+2}.$$

Because the map is independent of all the choices it then patches on affine open pieces and defines a map in the case V need not be affine.

Next use the map $L_i^{r+2} \rightarrow L_{i+1}$ to define a multiplication map $R_i R_j \rightarrow R_{i+j}$. For this we apply definition of the R_i in (2). For example if $r = 8$ Then $(r + 2) = 10$ and we are using familiar base ten expansions. If we wanted to calculate $98 + 3$ in the familiar way in base ten, we would ‘carry’ twice, writing

$$(9 \times 10 + 8) + 3 = (9 + 1) \times 10 + 1 = 10^2 + 1.$$

Correspondingly we define

$$R_{98} R_3 \rightarrow R_{101}$$

to be the composite

$$R_{98} R_3 = (L_1^9 L_0^8)(L_0^3) \rightarrow L_1^{10} L_0 \rightarrow L_2 L_0 = R_{101}$$

where the first map is induced by $L_0^{10} \rightarrow L_1$ and the second by $L_1^{10} \rightarrow L_2$.

To check the multiplication is well defined and satisfies the ring axioms it suffices to treat the affine case. Then R with its multiplication law has the structure explicitly described in the next section.
 QED

4. Elementary Construction of R .

When V is affine with coordinate ring R_0 , $\delta_1, \dots, \delta_r$ are a K basis of \mathcal{L} and $I \subset K$ is a fractional ideal we can define R in an elementary way to be the smallest subring of $K[T]$ such that

- i) R contains $R_0 \subset K$ in degree zero
- ii) Whenever the fractional ideal $IR = I \oplus IR_1 \oplus IR_2 \oplus \dots$ contains homogeneous elements f_0, \dots, f_r of the same degree i where i is either zero or equal to a possibly zero partial sum of the divergent geometric series

$$1 + (r + 2) + (r + 2)^2 + (r + 2)^3 + \dots$$

then R must contain the product Td where d is the determinant already displayed in (3) (now disregarding the phrase of text following the display).

2. Remark The construction of R appears mysterious for two reasons. Not only is the connection with the base $(r + 2)$ expansions unusual, also the choice of degrees for the grading is strange. Given that L_0, \dots, L_i have degrees $1, (r + 2), \dots, (r + 2)^i$ and $F(IL_0L_1 \dots L_i) = L_{i+1}$ has degree $(r + 2)^{i+1}$ we see from the formula

$$(r + 2)^{i+1} = 1 + (r + 1)(1 + (r + 2) + \dots + (r + 2)^i)$$

that the operation of F has the effect of multiplying degrees by $(r + 1)$ and then adding one. So the effect of F on degrees is neither addition or multiplication, but an affine transformation. This had mystified me for more than a year now, but it now seems the mystery will be solved in a later section of this paper, when we introduce separately a very ample divisor H and a canonical divisor K . What one shall see is the effect of multiplying an element in the span of these two divisors by the integer $r + 1$ and then adding K .

5. Statement of Results

3. Theorem Let V be an irreducible variety over a field k of characteristic zero. Let \mathcal{F} be a singular foliation on V . Let I be a sheaf of ideals on V . Let $V_0 \rightarrow V$ be blowing up I and subsequently let

$$\dots V_2 \rightarrow V_1 \rightarrow V_0 \rightarrow V$$

be the Gauss process for the foliation lifted to V_0 . The following are equivalent

1. The sheaf of graded \mathcal{O}_V algebras R defined using \mathcal{F} and I is finite type over k
2. The tower $\dots V_2 \rightarrow V_1 \rightarrow V_0 \rightarrow V$ stabilizes
3. The natural map $F(J)^{r+2} \rightarrow F(JF(J))$ is an isomorphism for some $J = IL_0 \dots L_{t-1}$ and some $t \geq 0$.

We will give two separate proofs of the theorem later on.

4. Corollary. A foliation on an irreducible variety V in characteristic zero can be resolved by a locally projective birational map (i.e., a blow up of a sheaf of ideals) if and only if there is an ideal sheaf I on V with $F(IF(I)) = F(I)^{r+2}$.

Proof of Corollary. Note that the words ‘‘ideal sheaf’’ can be replaced by ‘‘torsion free coherent sheaf of rank one’’ as any such sheaf can be embedded after twisting by a divisor. Assume there is an ideal sheaf I with $F(IF(I)) = F(I)^{r+2}$. Then [1] Theorem 15 part ii) with $N = 0$ shows that the Gauss process starting with blowing up I finishes in one further step. Conversely assume the foliation can be resolved and let V_0 be a resolution, with resolving ideal sheaf I . This time the Gauss process for the foliated variety V_0 finishes at step 0. Though there is no direct connection between the number of steps in the Gauss resolution (zero in this case) and the generating degrees of R , nevertheless, by part 1. of Theorem 3 we do know R is finite type. Then by part 3. of Theorem 3 the sheaf $J = IL_0 \dots L_{t-1}$ satisfies $F(JF(J)) = F(J)^{r+2}$ for some finite value of t . QED

As we explained in the introduction, one application is this

5. Corollary Let V be a normal foliated, irreducible projective variety over an algebraically closed field k of characteristic zero such

that $(r + 1)H + K$ has Iitaka dimension less than two, where H is a hyperplane section of V , r is the dimension of the foliation and K is the canonical divisor of the Foliation. Then the sequence of Gaussian blowups of V stabilizes.

Remark. Because the description of generating sections of $L_0 \subset \mathcal{O}_V(K + (r + 1)H)$ always includes a nonzero global section, There is always a choice of effective divisor linearly equivalent to $(r + 1)H + K$, and the Iitaka dimension is the transcendency degree of the algebra of global sections of the structure sheaf of the quasi projective variety $V \setminus D$.

6. Four affine examples.

1. Example. Firstly consider the cusp, whose coordinate ring is $k[x^2, x^3, x^4, \dots]$. The one dimensional foliation is spanned by any nonzero derivation so let's use $x\partial/\partial x$. Take I to be the unit ideal. Our ring R is now the smallest subring of $k[x, T]$ containing $k[x^2, x^3, \dots]$ and with the property that whenever R contains two monomials A, B whose x degrees are distinct and whose T degrees are the same, possibly zero, partial sum of the divergent geometric series $1 + 3 + 3^2 \dots$ then the product ABT must be contained in R . Here is therefore the list of monomials of low degree in R .

$$\begin{array}{cccccc}
 x^8 & Tx^8 & T^2x^8 & T^3x^8 & T^4x^8 & \\
 x^7 & Tx^7 & T^2x^7 & T^3x^7 & T^4x^7 & \\
 x^6 & Tx^6 & T^2x^6 & T^3x^6 & & \\
 x^6 & Tx^5 & T^2x^5 & T^3x^5 & & \\
 x^4 & Tx^4 & T^2x^4 & & & \\
 x^3 & Tx^3 & & & & \\
 x^2 & Tx^2 & & & &
 \end{array}$$

1

The monomials Tx^2 and T^3x^5 are the only interesting ones. They are the only nontrivial monomials in R which are not a product of monomials of smaller T degree. T^3x^5 is included because 1 is a partial sum of a geometric series of the base 3, and the separate monomials Tx^2 and Tx^3 have distinct x degrees, and so their product times T is included. The monomials in $L_i T^{3^i}$ are the columns in the diagram which are indexed by powers of three, and the last observation implies the ninth column is the third power of the third column. Then $L_1^3 = L_2$ so the desired equation $F(J)^3 = F(JF(J))$ holds for $J = IF(I)$. This example and the next one are included as contrasting tutorial examples.

2. Example. For a second example consider the singular foliation of the plane given by the vector field

$$x\partial/\partial x + 2y\partial/\partial y.$$

It is known that this foliation cannot be resolved by blowing up points, and that singular foliations of the plane can rarely be resolved by blowing up points. There is a general ‘desingularization theorem’ for one dimensional singular foliations of the plane, which for example is applied to complex singularities which are defined over the reals, as described in Ilyashenko’s centennial history of Hilbert’s 16’th problem [17]. According to Ilyashenko’s article, the desingularization theorem has a long history, with contributions by Bendixson, Seidenberg, Lefschetz, Dumortier, and van Essen, but is not written up in any book; it states that after blowing up points, a one dimensional foliation of the plane can be arranged to have isolated singular points with nonzero linear part, such that the linear part at each singular point has a nonzero eigenvalue. The question of which singular foliations can actually be resolved in the stronger sense of lifting to a nonsingular foliation after pulling back (and taking an irreducible component) by a locally projective birational morphism, even for linear foliations, is unsolved as far as I know and will not be solved in this paper, though we will consider which foliations can be resolved torically, and we expect there to be no surprises if resolutions which are not toric are allowed. If we apply our results to the case of linear plane foliations we will see that the linear foliations of the plane which can be resolved torically with respect to an eigenspace decomposition have the eigenvalue pairs $(1, 0)$, $(0, 1)$ and $(1, 1)$.

For the linear foliation of the plane we are considering now, with the eigenvalues 1 and 2, it is an easy calculation that the foliation cannot be resolved by blowing up points, and by Zariski [4] any proper birational map of normal surfaces arises that way. It is therefore not possible that this particular foliation could be resolved by a proper birational map from a normal surface, and this is easily seen directly. In this example and the next one, we will try to understand the deeper reason why it cannot be resolved, to motivate a more general theorem. In this example let us look at why the Gauss process itself does not converge. In view of the main theorem, it shall be the same thing to examine the ring R and verify that it is

not finitely generated. A k basis of R consists of the the smallest set of monomials in x, y, T which is closed under multiplication, such that all monomials $x^i y^j$ for $i, j \geq 0$ are included, and in addition if A, B are two monomials in x and y of distinct degree (where x is given degree 1 and y degree 2), and if AT^i and BT^i are included where i is a partial sum of the geometric series $1 + 3 + 3^2 + \dots$ then so is $T \cdot AT^i \cdot BT^i = T^{2i+1}AB$. Consider the monomials of the form $x^v y^j T^k$ which occur. For $v = 0$ we get $y^j T^k$ with $j \geq k \geq 0$. These arise as powers of y times powers of yT . For $v = 1$ we get the smallest set of monomials containing x and closed under multiplication by both y and yT and under

$$xy^j T^i \mapsto xy^{j+i} T^{2i+1}$$

when i is zero or of the form $1 + 3 + 3^2 \dots$. The latter rule comes from the product $T \cdot (m_0 T) \cdot (m_1 T)$ where $m_0 = y^i$ and $m_1 = xy^j$, as note that xy^j always has odd degree which is always distinct from that of y^i , which is even. Let $P(x, y, T)$ be the sum of all monomials in R . Then $\frac{\partial}{\partial x} P(x, y, t)|_{x=0}$ is the sum of the smallest set of monomials in y and T which contains all $y^{3^i - i - 1} T^{3^i}$ for $i = 0, 1, 2, \dots$ and is closed under multiplication by y and yT . This is

$$\frac{1}{1 - yT} \left(1 + \frac{yT}{y} + \frac{(yT)^3}{y^2} + \frac{(yT)^9}{y^3} + \frac{(yT)^{27}}{y^4} + \dots \right).$$

which is not a rational function so R is not finitely generated.

3. Example. For the third example, let us continue to consider the same foliation as in the previous example, but now let us try to understand why there is no toric resolution of this one-dimensional foliation. Our proof will introduce a technique which will be able to generalize to foliations of higher dimension. A toric resolution is a special case of blowing up a monomial ideal I . We shall show in fact that the foliation is not resolved by blowing up any monomial ideal, and in fact (which is equivalent) that it cannot be resolved by blowing up a monomial ideal and then following by a finite chain of Gauss blowups. Consider the infinite sequence of maps

$$\dots V_2 \rightarrow V_1 \rightarrow V_0 \rightarrow V$$

where $V_0 \rightarrow V$ is blowing up along I and $V_{i+1} \rightarrow V_i$ is the Gauss blowup along the foliation for $i \geq 1$. I claim the process can not converge, meaning none of the maps is an isomorphism. The theorem again says this is equivalent to finite generation of the appropriate ring R . Like in the previous example, finite generation of R would correspond to an equation $L_i^3 = L_{i+1}$. Writing $J = IL_0 \dots L_{i-1}$ we would have $F(J)^3 = F(JF(J))$. Let A be the smallest monomial in J for the alphabetic ordering (where x is given more priority than y). Let $B = Ay$ and $C = Ay^{-e}x$ with e maximum. The monomials AB and AC belong to $F(J)$ and so A^2B and A^2C belong to $JF(J)$. The degrees have opposite parity so are unequal, whence $(A^2B)(A^2C)$ belongs to $F(JF(J))$. The product cannot be rewritten any other way and it cannot belong to $F(J)^3$ because one of the three factors would have to be A^2 which does not belong to $F(J)$ at all.

4. Example. Our fourth example involves examining the conditions for resolvability of the unique codimension zero foliation on a normal irreducible affine complex algebraic surface. Taking affine coordinates x_0, \dots, x_n , we assume x_0 is the constant function $x_0 = 1$ and that x_1, x_2 are algebraically independent. Let y_0, \dots, y_m be rational functions and let $V_0 = Bl_I V$ where I is the fractional ideal generated by (y_0, \dots, y_m) .

We choose as our derivations $\partial/\partial x_1$ and $\partial/\partial x_2$. Since the characteristic of k is zero, and any rational function on V is algebraic over $k(x_1, x_2)$, and so the $\partial/\partial x_i$ for $i = 1, 2$ can be evaluated on any rational function, and thus define a pair of (commuting) derivations on the function field. For any fractional ideal I we may use these two derivations to view $F(I)$ again as a fractional ideal, as explained in section 4. The map $F(I)^{r+2} \rightarrow F(IF(I))$ is then an inclusion of fractional ideals, depending on the rational functions y_0, \dots, y_m .

Because the generating sequence of I is a product with the sequence x_0, \dots, x_n with $x_0 = 1$ the hypothesis of [1] Proposition 11 is satisfied and shows $F(I)$ is generated by determinants

$$\begin{vmatrix} x_e y_f & \partial/\partial x_1(x_e y_f) & \partial/\partial x_2(x_e y_f) \\ x_g y_h & \partial/\partial x_1(x_g y_h) & \partial/\partial x_2(x_g y_h) \end{vmatrix}$$

Since the generators of I satisfy the hypothesis of the proposition so do these generators times the determinants displayed above. Applying the same proposition to $IF(I)$ shows $F(IF(I))$ is generated by the determinants

$$\begin{vmatrix} x_a y_b \begin{vmatrix} x_c x_d & \partial/\partial x_1(x_c y_d) & \partial/\partial x_2(x_c y_d) \\ x_e y_f & \partial/\partial x_1(x_e y_f) & \partial/\partial x_2(x_e y_f) \\ x_g y_h & \partial/\partial x_1(x_g y_h) & \partial/\partial x_2(x_g y_h) \end{vmatrix} & \partial/\partial x_1 \left(\begin{vmatrix} x_a y_b & x_c y_d & \partial/\partial x_1(x_c y_d) & \partial/\partial x_2(x_c y_d) \\ x_e y_f & x_g y_h & \partial/\partial x_1(x_e y_f) & \partial/\partial x_2(x_e y_f) \\ x_i y_j & x_k y_l & \partial/\partial x_1(x_i y_j) & \partial/\partial x_2(x_i y_j) \\ x_m y_n & x_o y_p & \partial/\partial x_1(x_m y_n) & \partial/\partial x_2(x_m y_n) \end{vmatrix} \right) & \partial/\partial x_2 \left(\begin{vmatrix} x_a y_b & x_c x_d & \partial/\partial x_1(x_c y_d) & \partial/\partial x_2(x_c y_d) \\ x_e y_f & x_g y_h & \partial/\partial x_1(x_e y_f) & \partial/\partial x_2(x_e y_f) \\ x_i y_j & x_k y_l & \partial/\partial x_1(x_i y_j) & \partial/\partial x_2(x_i y_j) \\ x_m y_n & x_o y_p & \partial/\partial x_1(x_m y_n) & \partial/\partial x_2(x_m y_n) \end{vmatrix} \right) \\ x_i y_j \begin{vmatrix} x_k x_l & \partial/\partial x_1(x_k y_l) & \partial/\partial x_2(x_k y_l) \\ x_o y_p & \partial/\partial x_1(x_o y_p) & \partial/\partial x_2(x_o y_p) \\ x_q y_s & \partial/\partial x_1(x_q y_s) & \partial/\partial x_2(x_q y_s) \end{vmatrix} & \partial/\partial x_1 \left(\begin{vmatrix} x_i y_j & x_k x_l & \partial/\partial x_1(x_k y_l) & \partial/\partial x_2(x_k y_l) \\ x_m y_n & x_o y_p & \partial/\partial x_1(x_o y_p) & \partial/\partial x_2(x_o y_p) \\ x_q y_s & x_r y_t & \partial/\partial x_1(x_q y_s) & \partial/\partial x_2(x_q y_s) \end{vmatrix} \right) & \partial/\partial x_2 \left(\begin{vmatrix} x_i y_j & x_k x_l & \partial/\partial x_1(x_k y_l) & \partial/\partial x_2(x_k y_l) \\ x_m y_n & x_o y_p & \partial/\partial x_1(x_o y_p) & \partial/\partial x_2(x_o y_p) \\ x_q y_s & x_r y_t & \partial/\partial x_1(x_q y_s) & \partial/\partial x_2(x_q y_s) \end{vmatrix} \right) \\ x_t y_u \begin{vmatrix} x_c x_d & \partial/\partial x_1(x_c y_d) & \partial/\partial x_2(x_c y_d) \\ x_e y_f & \partial/\partial x_1(x_e y_f) & \partial/\partial x_2(x_e y_f) \\ x_g y_h & \partial/\partial x_1(x_g y_h) & \partial/\partial x_2(x_g y_h) \end{vmatrix} & \partial/\partial x_1 \left(\begin{vmatrix} x_t y_u & x_c x_d & \partial/\partial x_1(x_c y_d) & \partial/\partial x_2(x_c y_d) \\ x_e y_f & x_g y_h & \partial/\partial x_1(x_e y_f) & \partial/\partial x_2(x_e y_f) \\ x_i y_j & x_k y_l & \partial/\partial x_1(x_i y_j) & \partial/\partial x_2(x_i y_j) \\ x_m y_n & x_o y_p & \partial/\partial x_1(x_m y_n) & \partial/\partial x_2(x_m y_n) \end{vmatrix} \right) & \partial/\partial x_2 \left(\begin{vmatrix} x_t y_u & x_c x_d & \partial/\partial x_1(x_c y_d) & \partial/\partial x_2(x_c y_d) \\ x_e y_f & x_g y_h & \partial/\partial x_1(x_e y_f) & \partial/\partial x_2(x_e y_f) \\ x_i y_j & x_k y_l & \partial/\partial x_1(x_i y_j) & \partial/\partial x_2(x_i y_j) \\ x_m y_n & x_o y_p & \partial/\partial x_1(x_m y_n) & \partial/\partial x_2(x_m y_n) \end{vmatrix} \right) \end{vmatrix}$$

The equation $F(I)^4 = F(IF(I))$ holds just when the large determi-

nants can be expressed as homogeneous polynomials of degree four in the small determinants. Because V is normal, the coefficients of the degree four polynomials whenever they exist would restrict to well-defined functions on the regular locus of V . Likewise, all y_i in the equation can be multiplied by a common denominator t of the coordinate ring, both sides of the equation are multiplied by $t^{(r+1)(r+2)}$. Then too the y_i may be considered as well-defined functions on the whole of the regular locus of V . The equation is therefore equivalent to a system of partial differential equations on a complex manifold. This is the system of differential equations one would have encountered in the real case if one were to have looked for an algebraic condition which bounds the Gaussian curvature of the leaves under a projective embedding. A solution (y_0, \dots, y_m) need not generate a resolving ideal, nor does a set of generators of a resolving ideal necessarily provide a solution. Zariski's work [3] showing surfaces can be resolved does provide a resolving ideal. The chain of subsequent Nash blowups finishes because it is trivial. The ring R is an algebraic model of the chain, which however need not become trivial at the first step. Theorem 3 part 1 shows R is finite type. Then there is an ideal J satisfying part 3 of the same theorem. Taking y_0, \dots, y_m to be a generating sequence we see therefore that this, possibly familiar, system of differential equations on the smooth manifold of V always has a solution.

7. Foliations coming from split semisimple representations of commutative Lie algebras.

The simplest examples of singular foliations arise from a faithful split semisimple representations of an r dimensional commutative Lie algebra G on a vector space V (all over the base field k of characteristic zero). We choose coordinates x_1, \dots, x_n on V and elements in the dual of G $\alpha_1, \dots, \alpha_n \in \widehat{G}$ so that for $s \in G$ we have

$$sx_i = \alpha_i(s)x_i.$$

We may view the vector space V as a toric variety such that the monomials in x_i are the characters of the torus which are well-defined on all of V .

6. Corollary. It is possible to resolve the foliation on V torically if and only if the underlying set of nonzero roots (counted without multiplicities) forms a basis of the dual Lie algebra.

7. Remarks. The proof shows more. It allows a more general blowup of an arbitrary monomial ideal. Also the theorem applies in cases when the blowup may be a singular variety.

Proof. For any monomial ideal I view $F(I)$ as the ideal generated by determinants of matrices whose rows are the $(f, \delta_1 f, \dots, \delta_r f)$ where $f, g \in I$. By multilinearity of the determinant, we may restrict f to run over monomials in I , and so again $F(I)$ is a monomial ideal. In this way we inductively see that all L_i are monomial ideals.

Suppose first that the foliation can be resolved by some monomial ideal I then Theorem 3 implies that $F(J)^{r+2} = F(JF(J))$ for some $J = IL_0 \dots L_{t-1}$, again a monomial ideal. Let A be the monomial in J which is minimum for the alphabetical ordering on (i_1, \dots, i_n) where i_1 is given least significance. Also let B be the smallest monomial in J in which the power of x_{r+1} is one larger than what occurs in A .

Since the V is a faithful representation of our Lie algebra G , we know there must exist a basis of \widehat{G} consisting of roots. By choice of numbering these may be assumed to be $\alpha_1, \dots, \alpha_r$. If $n = r$ we are done, so assume $n \geq r + 1$.

For any monomial $M = x_1^{i_1} \dots x_n^{i_n}$ write $f(M) = i_1 \alpha_1 + \dots + i_n \alpha_n$. In this way we obtain a homomorphism from the set of monomials (characters of the torus that are well defined on all of V) to the dual of G . Consider the product $A^{(r^2+3r+1)}(x_1 \dots x_{r-1})^{r+2} x_r^r B$. First we associate this as a product of $r + 1$ factors

$$\left(\prod_{i=1}^{r-1} [x_i A^{r+2}(x_1 \dots x_r)] \right) \cdot [A^{r+2}(x_1 \dots x_r)] \cdot [A^{r+1} x_1 \dots x_{r-1} B]$$

Let us show that each factor is a product of $r + 2$ monomials with f values which span \widehat{G} affinely. The first factor is

$$A \cdot A \cdot Ax_1^2 \cdot Ax_2 \cdot \dots \cdot Ax_r$$

and applying f to each term and subtracting $f(A)$ yields

$$0, 0, 2\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_r$$

which affinely span since the $\alpha_1, \dots, \alpha_r$ are linearly independent. All but the last factor behave similarly to this one with the coefficient of 2 occurring in a different position or being absent. The last factor is $A \cdot A \cdot Ax_1 \cdot \dots \cdot Ax_{r-1} \cdot B$. This time after subtracting $f(A)$ from each term, the sequence of f values is $0, 0, \alpha_1, \dots, \alpha_{r-1}, f(B) - f(A)$. Modulo the hyperplane spanned by the other terms, the last term is congruent to $\alpha_{r+1} - i\alpha_r$ for some positive integer i . We shall aim to prove this is congruent to zero for some positive value of i . Since this deduction will be true after any permutation of the variables it will be a strong restriction, which will imply our conclusion. In the way of hoping for a contradiction, then, suppose that there is no positive number i so that α_{r+1} is congruent to $i\alpha_r$ modulo the span of $\alpha_1, \dots, \alpha_{r-1}$. Then the final factor too is a product of $(r + 1)$ monomials in J whose images in \widehat{G} are affinely independent.

Next let us apply f to each of the $(r + 1)$ whole factors in square brackets. Subtracting $(r+2)f(A) + \alpha_1 + \dots + \alpha_r$ from each we obtain the sequence

$$\alpha_1, \alpha_2, \dots, \alpha_{r-1}, 0, f(B) - f(A) - \alpha_r.$$

The last term is congruent to $\alpha_{r+1} - (i + 1)\alpha_r$ and again using our assumption that this is not in the span of $\alpha_1, \dots, \alpha_{r-1}$ we have a sequence that is affinely independent. This establishes that our monomial belongs to $F(JF(J))$.

Because of our assumption that $F(JF(J)) = F(J)^{r+2}$ our monomial must then belong to $F(J)^{r+2}$. This means it must be possible to refactorize our product a different way, as a product of $(r + 2)$ monomials in J so that each monomial factorizes as a product of $r + 1$ monomials with affinely independent images in \widehat{G} . The most significant letter where our product differs from $A^{(r+2)(r+1)}$ is in the exponent of x_{r+1} , which is one larger. In our factorization as a product of $(r + 2)(r + 1)$ monomials in J , choose one of the monomials which includes a higher power of x_{r+1} than A does. Remove this factor from our product expression, along with whichever r further factors are associated with it. At most r of the remaining $(r + 1)^2$ factors can have a power of x_r that is larger than what occurs in A . Applying f to each factor we obtain a sequence of $(r + 1)^2$ elements of \widehat{G} of which at least $r^2 + r + 1$ lie in an affine hyperplane, let us call it H . The sequence is a disjunction of $r + 1$ subsequences, each of which consists of $(r + 1)$ affinely independent elements. An affinely independent set always contains an element in $\widehat{G} \setminus H$ and so there are at least $r + 1$ terms of our sequence in $\widehat{G} \setminus H$. This is the desired contradiction.

We have shown, therefore, that for every set of roots $\alpha_1, \dots, \alpha_{r+1}$ such that $\alpha_1, \dots, \alpha_{r-1}$ are linearly independent, it must be the case that $\alpha_r - i\alpha_{r+1}$ lies in the linear span of $\alpha_1, \dots, \alpha_{r-1}$ for some integer $i \geq 0$. Interchanging r and $r + 1$ we see that this is true for $i = 1$. Letting α_r and α_{r+1} range over all roots not equal to $\alpha_1, \dots, \alpha_{r-1}$ we see that all roots belong to the hyperplane spanned by $\alpha_1, \dots, \alpha_{r-1}$ together with one affine translate of that hyperplane. Applying this principle to a basis $\alpha_1, \dots, \alpha_r$ of \widehat{G} we see that all roots are of the form $a_1\alpha_1 + \dots + a_r\alpha_r$ for $a_i \in \{0, 1\}$. So any two sets of roots which are vector space bases of \widehat{G} give rise to a change-of-basis matrix with positive integer entries. But the only invertible elements in the monoid of matrices with positive integer entries are permutation matrices. QED

8. Proof of Theorem 3.

In this paper we will give two proofs of the theorem. The first proof in this section does not use any algebraic geometry. A subsequent proof in section 11 will be outlined which is simpler but uses concepts and theorems of algebraic geometry.

Let us first treat the affine case. Let V be an irreducible affine variety over a field k of characteristic zero with function field K , $\mathcal{L} \subset \mathcal{D}er_k(K, K)$ a K linear Lie algebra, with fixed basis $\delta_1, \dots, \delta_r$. We will call the coordinate ring of V R_0 in the expectation that it will later become the zero degree component of a graded ring R . Most simply, the torsion free R_0 modules $L_i, J_i, R_i \subset K$ can be identified with the fractional ideals ($= R_0$ modules) defined by (1), (2), and (3) where in (3) one may take f_i to run over all elements of I .

Step 1: Let us prove $1. \Rightarrow 3 \Rightarrow 2.$ in Theorem 3. Thus suppose 1. We suppose R is finite type. By [1] Lemma 4 and induction, the resolution process $\dots V_2 \rightarrow V_1 \rightarrow V_0 \rightarrow V$ has the property that $V_i = Bl_{J_i}(V)$ where we recall $J_i = IL_0 \dots L_{i-1}$. By construction, R is generated by R_0 and the L_i in degree $(r+2)^i$ for $i = 0, 1, 2, \dots$. By very elementary properties of integer base $(r+2)$ expansions, the only way R can be finite type is if actually $L_{i+1} = L_i^{r+2}$ for some i . This proves part 3. Now by [1] Lemma 4 and induction, the resolution process $\dots V_2 \rightarrow V_1 \rightarrow V_0 \rightarrow V$ has the property that $V_i = Bl_{J_i}(V)$ where we recall $J_i = IL_0 \dots L_{i-1}$. This implies blowing up J_{i+1} or J_{i+2} has the same effect so $V_{i+2} = V_{i+1}$ and part 2. is proved.

The remaining steps will be concerned with proving $2. \Rightarrow 1.$ in Theorem 3. From now on we will assume the resolution process finishes.

Step 2: In this step we will define a subring of R . We are assuming that resolution process finishes in finitely many steps. We know by [1] Theorem 15 (ii) for some N

$$J_i^N L_i^{r+2} \rightarrow J_i^N L_{i+1} \quad (4)$$

is an isomorphism.

Remark. We alternatively could now deduce this for a possibly smaller value of i , using the fact that eventually the Atiyah class

on V_i becomes “multiplicatively trivial” in the sense that the tensor product of the highest exterior powers of the end terms of the Atiyah sequence maps isomorphically to the highest exterior power of the middle term. The fact that we know the sheaves become the same after multiplying by a power of J_i is related to the fact that the Atiyah classes are calculated on the V_i and not V . The rough idea can be summarized as saying now that finite generation questions on V_i and on V should be equivalent because coherence is preserved under direct images of a proper map.

Fix the values of i and N for the rest of the proof.

Let the graded subring $R^{[i+1]} \subset R$ be the subring which is obtained by multiplying each degree j component R_j by $(IL_0L_1\dots L_i)^j = J_{i+1}^j$. We have

$$R^{[i+1]} = \bigoplus_j J_{i+1}^j R_j.$$

Before we prove R is finite type over k we will first prove the subring $R^{[i+1]}$ is finite type over k . Our proof will implicitly determine bounds for the degrees of the generators of $R^{[i+1]}$.

Step 3: In this step we will prove a lemma which will later be useful in proving that the subring $R^{[i+1]}$ is finite type.

8. Lemma For any $s \geq 0$

$$J_{i+1}^{N(r+1)^s} L_{i+s+1} = J_{i+1}^{N(r+1)^s} L_i^{(r+2)^{s+1}}.$$

Proof. If $s = 0$ multiply the previous formula (4) by L_i^N and use $J_{i+1} = L_i J_i$. If $s \geq 1$ assume the lemma true for smaller s and we have that the left hand side equals

$$\begin{aligned} & J_{i+1}^{[N(r+1)^{s-1}-1](r+1)+(r+1)} L_{i+s+1} \\ &= J_{i+1}^{[N(r+1)^{s-1}-1](r+1)+(r+1)} F(IL_0 L_1 \dots L_{i+s}) \end{aligned}$$

Using Theorem 12 of [15] which implies $(J^{r+1})F(X) \subset F(JX)$ for any X we find this is

$$\subset J_{i+1}^{r+1} F(J_{i+1}^{N(r+1)^{s-1}-1} L_0 \dots L_{i+s})$$

Recall $J_{i+1} = IL_0 \dots L_i$ so we know $J_{i+1}^{N(r+1)^{s-1}-1} IL_0 \dots L_i = J_{i+1}^{N(r+1)^{s-1}}$. The inductive hypothesis implies $J_{i+1}^{N(r+1)^{s-1}} L_{i+\alpha} = J_{i+1}^{N(r+1)^{s-1}} L_i^{(r+2)^\alpha}$ for $\alpha = 0, 1, \dots, s$. Simplifying the expression above using both rules yields

$$= J_{i+1}^{r+1} F((IL_0 \dots L_i)^{N(r+1)^{s-1}} L_i^{(r+2)+(r+2)^2+\dots+(r+2)^s}).$$

This is of the form $J_i^{r+1} L_i^{r+1} F(J_i^a L_i^b)$ for some numbers a and b and here the way we proceed depends on whether a and b are odd or even. If they are both even we use the fact that $F(X^2) = X^{r+1} F(X)$ for any X , which is a special case of [1] Theorem 14, while if for example a is odd and b is even we first apply the inclusion $\subset L_i^{r+1} F(J_i^{a+1} L_i^b)$ and then apply the rule $F(X^2) = X^{r+1} F(X)$. In this way we obtain

$$\subset J_i^{N(r+1)^s} L_i^{(r+1)+[N(r+1)^{s-1}+(r+2)+\dots+(r+2)^{s-1}](r+1)} F(J_i L_i).$$

But $F(J_i L_i) = F(J_{i+1}) = L_{i+1}$ can be replaced by L_i^{r+2} because it occurs multiplied by at least J_{i+1}^N and we obtain

$$= J_{i+1}^{N(r+1)^s} L_i^{(r+2)^{s+1}}$$

as claimed.

Step 4: In this step let us just observe that it follows from lemma 8 by letting $t = s + i + 1$ that for t sufficiently large

$$\begin{aligned} J_{i+1}^{(r+2)^t} L_t &= J_{i+1}^{(r+2)^t} L_i^{(r+2)^{t-i}} \\ &= (J_{i+1}^{(r+2)^i} L_i)^{(r+2)^{t-i}} \end{aligned}$$

This is because then the inequalities

$$\left(\frac{r+2}{r+1}\right)^t \geq N.$$

and

$$s \geq 0$$

will both hold.

Step 5. Now we can show that the subring $R^{[i+1]} \subset R$ defined at the beginning of the proof is finite type. If the base $r + 2$ expansion of t is

$$a_0 + a_1(r+2) + \dots + a_m(r+2)^m$$

then the degree t component of $R^{[i+1]}$ is

$$R_t^{[i+1]} = (J_{i+1} L_0)^{a_0} \dots (J_{i+1}^{r+2} L_1)^{a_1} (J_{i+1}^{(r+2)^2} L_2)^{a_2} \dots (J_{i+1}^{(r+2)^m} L_m)^{a_m}$$

and so step 4 shows $R^{[i+1]}$ is generated by the $J_{i+1}^{(r+2)^t} L_t$ for

$$t < \max\left[i+1, \frac{\log(N)}{\log(r+2) - \log(r+1)}\right].$$

Step 6. Now we can deduce that R itself is finite type. This step does not give explicit bounds as it is based on Hilbert's basis theorem. To simplify notation, let us rename the graded ring $R^{[i+1]}$ by the name W . Since W is finite type over k and therefore over R_0 it has a sequence of homogeneous generators x_1, \dots, x_t of positive degree. Let d be the least common multiple of the $\text{degree}(x_i)$ and consider all possible elements of W of degree d which occur as monomials in the x_i . These monomials generate the 'truncated' ring $\bigoplus_j W_{dj}$. The fact that all generators belong to W_d implies

$$W_{dj} = W_d^j$$

Using the definition of $W = R^{[i+1]}$ this tells us that the inclusion

$$R_d^j \subset R_{dj}$$

becomes an equality after multiplying by $(IL_0 \dots L_i)^{dj}$. Take $x \in R_{dj}$ and let e_1, \dots, e_n be a set of R_0 module generators of $(IL_0 \dots L_i)^{dj}$. We have $xe_i = \sum a_{ij}e_j$ with $a_{ij} \in R_d^j$. We now apply Zariski and Samuel's trick of taking determinant of the matrix $(a_{ij} - x\delta_{ij})$. This gives $0 = x^n x^{n-1} b_{n-1} + \dots + b_0$ with the $b_t \in R_d^{j(n-t)}$. We view this as an equation of integral dependence where the coefficients b_t lie in the graded ring $\oplus_i R_d^i$ and this shows every element of R whose degree is a multiple of d is integral over this ring. Since R_d is finitely generated as R_0 module and R_0 is finite type, the ring $\oplus_i R_d^i$ is finite type over k and by the theorem of 'finiteness of normalization' we know the truncation $\oplus_i R_{id}$ since it consists only of integral elements, is also finite over $\oplus_i R_d^i$. Therefore the truncation is finite type over k and therefore Noetherian. Finally, the ring R is isomorphic to a direct sum of d ideals over the truncated ring (multiplying by any element of degree congruent to $-i \pmod d$ gives an embedding of the sum of terms congruent to $i \pmod d$ into the truncated ring). Since ideals in a Noetherian ring are finitely generated, we conclude R is finite over the truncated ring. It follows then that R is finite type over k .

Step 7. The proof in the affine case is done, but we have to consider the case V is not affine. Then more functorial definition of R shows it patches on a suitable affine open cover. QED

9. Discussion

Although the proof of Theorem 3 was algebraic, it might be helpful to visualize the corresponding diagram involving the prescheme $X = Proj(R)$, which does exist even when R is not finite type. Letting $V_0 = Bl_I(V)$ and letting $\dots V_2 \rightarrow V_1 \rightarrow V_0$ be the Gauss process for this variety, then writing $X_i = Proj(R^{[i]})$ we have a pullback diagram

$$\begin{array}{ccccccc} \dots & \rightarrow & X_2 & \rightarrow & X_1 & \rightarrow & X_0 & \rightarrow & X \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \dots & \rightarrow & V_2 & \rightarrow & V_1 & \rightarrow & V_0 & \rightarrow & V \end{array}$$

If the Gauss process for $V_0 = Bl_I(V)$ converges then for some n we have $V_{n+1} = V_n$ and $X_n = X_{n-1}$. Then the vertical map $X_n \rightarrow V_n$ is an isomorphism showing V_n is the pullback of $V_{n-1} \rightarrow V \leftarrow X$. Thus X is dominated by a variety.

10. Algebraic geometry interpretation

Let V is a normal irreducible quasiprojective variety over a field k of characteristic zero, and let H be a very ample divisor on V . Choose a singular foliation of dimension r . We shall consider the sheaf $\mathcal{O}_V(H)$ and the graded sheaf $R(\mathcal{O}_V(H))$.

9. Theorem. $R_i(\mathcal{O}_V(H))$ occurs as a subsheaf of $\mathcal{O}_V(irH + iH + iK_V)$ generated by global sections.

As an incidental remark, it can be proven from the definitions that $R(\mathcal{O}_V(H))$ is just the same as R up to twisting, such that $R_i(\mathcal{O}_V(H))$ can be identified with $\mathcal{R}_i(i(r+1)H)$.

Proof of Theorem 9. Let us write the determinant in question in the form which resembles the differential in the Beilinson resolution [9]. For any divisor E and global sections f_0, \dots, f_r of $\mathcal{O}_V(E)$ we can construct a global section $w(f_0, \dots, f_r)$ of $\mathcal{O}_V((r+1)E + K)$ by the formula

$$w(f_0, \dots, f_r) = \sum_{i=0}^r (-1)^i f_i df_0 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_r$$

where the hat over the factor df_i indicates that this factor is to be deleted. We will now state and prove a lemma and then afterwards

give the proof of Theorem 9.

10. Lemma The expression $w(f_0, \dots, f_r)$ is ‘homogeneous’ in the sense that if u is any rational function

$$w(uf_0, \dots, uf_r) = u^{r+1}w(f_0, \dots, f_r).$$

Proof of Lemma 10. The left side is the right side plus

$$\begin{aligned} u^r \cdot \sum_{i=0}^r (-1)^i f_i \left(\sum_{j=0}^{i-1} f_j df_1 \wedge df_2 \wedge \dots \wedge du \wedge \dots \wedge df_{i-1} \wedge df_{i+1} \wedge \dots \wedge df_r \right. \\ \left. + \sum_{j=i+1}^r f_j df_1 \wedge \dots \wedge df_{i-1} \wedge df_{i+1} \wedge \dots \wedge du \wedge \dots \wedge df_r \right). \end{aligned}$$

Here the term du replaces df_j .

There are two cancelling terms in which each pair of symbols df_i and df_j is removed.

Proof of Theorem 9. In Lemma 10, observe that if one of the f_i is taken to be 1 then the expression gives the wedge product of the remaining f_i . If the f_i are allowed to range over local sections of \mathcal{O} the $w(f_0, \dots, f_r)$ span a copy of the highest exterior power of the differentials of V modulo torsion. Whereas, if the f_i range over local sections of $\mathcal{O}_V(E)$ then the $w(f_0, \dots, f_r)$ generate a copy of the highest exterior power of the differentials, twisted by $(r+1)E$.

If \mathcal{F} is any subsheaf of $\mathcal{O}_V(E)$ and if the f_i are allowed to range over local sections but required to belong to the subsheaf \mathcal{F} , then the $w(f_0, \dots, f_r)$ generate a copy of $F(\mathcal{F})$ where F is the functor we used above, in the inductive definition of the L_i .

Thus we see

$$\mathcal{F} \subset \mathcal{O}_V(E) \Rightarrow F(\mathcal{F}) \subset \mathcal{O}_V((r+1)E + K)$$

Assuming inductively we have represented

$$L_j(\mathcal{O}_V(H)) \subset \mathcal{O}_V((r+2)^j rH + (r+2)^j H + (r+2)^j K_V)$$

for $j \leq i$. Consider the product sheaf

$$\mathcal{O}_V(H)L_0(\mathcal{O}_V(H))\dots L_i(\mathcal{O}_V(H)).$$

This – the argument to which F is to be applied – is a subsheaf of

$$\begin{aligned} & \mathcal{O}_V(H+(r+1)(1+(r+2)+(r+2)^2+\dots+(r+2)^i)H+(1+(r+2)+\dots+(r+2)^i K) \\ &= \mathcal{O}_V(H + ((r+2)^{i+1} - 1)H + \frac{((r+2)^{i+1} - 1)}{r+1} K). \end{aligned}$$

We have seen that F multiplies both degrees by $(r+1)$ and adds K , so we can represent L_{i+1} in

$$\mathcal{O}_V((r+1)(r+2)^{i+1}H + (r+2)^{i+1}K)$$

as needed.

Now since $R_i(\mathcal{O}_V(H))$ is a product of the L_i for various values of i we obtain the result desired embedding.

Finally, now, we need to verify that the sections we have used in the proof can be taken to be global. Or, rather, that the global sections which we have described generate the correct sheaves.

The issue is this. Choosing an affine open set $U \subset V$ we know the $w(f_0, \dots, f_r)$ generate $F(\mathcal{F})(U)$ if the f_i range over the full module of sections $\mathcal{F}(U)$. But it is not necessarily true – and easy to find cases when it is false – that the $w(f_0, \dots, f_r)$ generate $F(\mathcal{F})(U)$ if all we know is that the f_i range over a sequence of $\mathcal{O}_V(U)$ -module generators of $\mathcal{F}(U)$. In other words, we may not merely state that the argument we have given works when restricted to generating subsets.

However, we do know from [1] Proposition 11 that if we choose a sequence of k algebra generators of $\mathcal{O}_V(U)$, x_0, x_1, \dots, x_n say, such that $x_0 = 1$, then $F(\mathcal{F})(U)$ will be generated by the $w(g_0, \dots, g_r)$ where the g_i range over the pairwise products $x_i f_j$.

Now, at any time when we are applying F we apply it to a product $\mathcal{O}(H)L_0(\mathcal{O}(H))\dots L_i(\mathcal{O}(H))$ in which one of the factors is $\mathcal{O}(H)$. If we inductively assume all $L_i(\mathcal{O}(H))$ are generated by global sections we have specified already, and if we specify x_0, \dots, x_n to be a spanning sequence of global sections of $\mathcal{O}_V(H)$, then what we shall do is identify V with the quasi-projective variety one obtains using the embedding via the linear system coming from H . Thus V is covered by standard open subsets each of which is isomorphic to a subset of affine space with the coordinates x_0, \dots, x_n in which some x_i is

equal to one. We shall choose such a standard open set, and choose the numbering so that $i = 0$. Now, the global sections of the product sheaf which arise as $i + 1$ fold products of the chosen sections on U of the separate sheaves include all the products of the global sections of the $L_j(\mathcal{O}_V(H))$ which we have specified, ie times the section $x_0 = 1$ of the leftmost sheaf, and also the same sections multiplied by each of x_1, \dots, x_n .

Therefore upon applying the action of F as a functor operating on sheaves over this affine variety, the resulting sheaf $L_{i+1}(\mathcal{O}_V(H))$ is indeed generated by the restrictions of the sections $w(f_0, \dots, f_r)$ to our affine open subset, where f_0, \dots, f_r range over the specified products of global sections.

This proves that the global sections produced by this process restrict to module generators on each open subset of the standard open cover of V .

And therefore that they generate the sheaves $R_i(\mathcal{O}_V(H))$

We also see, taking T to be a fixed section of $\mathcal{O}_V((r+1)H)$, that the direct sum of the subsheaves of the $\mathcal{O}_V(irH + iH + K_V)$ is closed under multiplication, and in fact it is the same sheaf of rings we have constructed already in earlier sections on the affine open set where T is not zero.

11. Algebraic geometry proof of the theorem.

Now we give a quite easy algebraic geometry proof of theorem 3. We shall give only the important part of the proof, in the most important case. Thus we assume V is normal. We assume we are dealing with the codimension zero foliation only. These are merely simplifying restrictions. And we give only the implication which is most difficult, so we assume the chain of Nash blowups $\dots \rightarrow V_{i+1} \rightarrow V_i \rightarrow \dots \rightarrow V_0 \rightarrow V$ stabilizes where $V_0 \rightarrow V$ is blowing up a sheaf of ideals I , and we shall prove the graded sheaf R is finite type.

Assume then that for some i the map $V_i \rightarrow V_{i-1}$ is an isomorphism. Define divisors E, K_0, K_1, \dots on V_i such that $\mathcal{O}_{V_i}(E)$ is the pullback to V_i of I (so E is the negative of an exceptional Cartier divisor) and $\mathcal{O}_{V_i}(K_j)$ is the pullback to V_i of the highest exterior power of the differentials of V_j mod torsion. Note K_j has been defined even

for values of j larger than i since $V_i = V_{i+1} = V_{i+2} \dots$

Let π be the composite map $V_i \rightarrow V$. We shall prove the following lemma by induction.

11. Lemma. For all $j \geq 0$

$$\pi^* L_j(I) = \mathcal{O}_{V_i}(X_j)$$

where X_j is the divisor

$$X_j = K_j + (r+1)[(r+2)^j E + \sum_{t=0}^{j-1} (r+2)^{j-1-t} K_t].$$

Proof. Write

$$L_j(I) = F(IL_0(I)L_1(I)\dots L_{j-1}(I)).$$

By [1] corollary 3

$$\pi^* F(IL_0(I)\dots L_{i-1}(I)) \cong \pi^*(IL_0(I)\dots L_{i-1}(I))^{r+1} \Lambda^r \Omega \cong \mathcal{O}_{V_i}((r+1)(E + \sum_{s=0}^{i-1} X_s) + K_i).$$

By the inductive hypothesis (the statement of the lemma applied to each X_s) this equals \mathcal{O}_V twisted by

$$\begin{aligned} & (r+1)(E + \sum_{s=0}^{i-1} (K_s + (r+1)[(r+2)^s E + \sum_{t=0}^{s-1} (r+2)^{s-1-t} K_t]) + K_i \\ &= (r+1)(1+(r+1) \sum_{s=0}^{i-1} (r+2)^s) E + (r+1) \sum_{s=0}^{i-1} K_s + \sum_{t=0}^{i-1} \sum_{s=t+1}^{i-1} (r+1)^2 (r+2)^{s-1-t} K_t + K_i \\ &= (r+1)(1+(r+2)^i - 1) E + (r+1) \sum_{t=0}^{i-1} ((r+2)^{i-1-t} - 1) K_t + (r+1) \sum_{t=0}^{i-1} K_t + K_i \\ &= (r+1)(r+2)^i E + (r+1) \sum_{t=0}^{i-1} (r+2)^{i-1-t} K_t + K_i \end{aligned}$$

as required. QED

Now we resume the proof that $R(I)$ is finite type. Since $V_i = V_{i-1}$ then $K_i = K_{i-1}$. The formula just proven with K_i replaced by K_{i-1} shows

$$X_i = K_{i-1} + (r+1)(r+2)^i E + (r+1)(r+2)^{i-1} K_0 + \dots + (r+1)K_{i-1}.$$

Combining the terms K_{i-1} and $(r+1)K_{i-1}$ we obtain

$$\begin{aligned} &= (r+2)(K_{i-1} + (r+1)(r+2)^{i-1} E + (r+1)(r+2)^{i-2} K_0 + \dots + (r+1)K_{i-2}) \\ &= (r+2)X_{i-1}. \end{aligned}$$

The graded sheaf

$$\pi^* R(I)$$

in dimension j when the base $r+2$ expansion of j is $j = a_0 + a_1(r+2) + \dots + a_s(r+2)^s$, is just

$$\mathcal{O}_V(a_0 X_0 + \dots + a_s X_s).$$

Since $X_{j+1} = (r+2)X_j$ for large j , the positive linear combinations of all the divisors X_0, X_1, \dots , are spanned by finitely many of the X_i . In other words the X_j generate a finitely generated monoid in the Weil divisor group. It follows that $\pi^* R(I)$ is finite type over k . The pushforward of each term

$$\pi_* \pi^* R_j(I)$$

is the integral closure of $R_j(I)$ by [11] page 219. This is because we are considering the codimension zero foliation so V_i is actually a nonsingular variety for large i . Actually [L] refers to sheaves of ideals but the theorem applies to torsion free rank one coherent sheaves, and the notion of integral closure extends. Moreover $\pi_* \pi^* R(I)$ is contained in the integral closure of the sheaf of rings $R(I)$. Applying the result which states that an algebra over k whose integral closure is finite type over k is itself finite type over k , this shows that the part of $R(I)$ lying over each affine open set is a graded k algebra of finite type. This finishes the alternative, algebraic geometry, proof of the more difficult implication of the theorem, subject to the inessential simplifying assumption that V is normal and the foliation is codimension zero. QED

12. A final result.

We finish with a final result. Let V be a normal irreducible complex projective variety with a resolvable foliation, and H a very ample divisor. Our hypothesis implies that there must exist a vector subspace $X \subset H^0(V, \mathcal{O}_V(iH))$ for some value of i so that blowing up the base locus of X resolves the singularities of V . We may view the elements of X as homogeneous polynomials of degree i .

12. Theorem. There are functorial locally closed conditions on the vector subspaces $X \subset |iH|$, not vacuous for all i , which when true, ensure that blowing up the base locus of X and one further Gaussian blowup resolves the foliation.

Remark. In case of the codimension zero foliation, which is always resolvable (since singularities can be resolved) the conditions ensure that the blowup of the base locus of X is an immersed image of a smooth manifold. This is because the map $L_0^{r+2} \rightarrow L_1$ pulled back first to V_0 , the blowup of I , and then pulled back further to V_1 , the Nash blowup of that, yields up to twisting (the same on both sides), the map from the pullback of the r 'th exterior power of the differentials of V_0 modulo torsion, to the same for V_1 . This being surjective implies that V_0 itself is an immersed image of a smooth complex manifold.

Remark. The reason for the word ‘functorial,’ which we will not define in this context precisely, is that, without some modification the theorem is vacuously true. One needs to rule out proofs such as, to just arbitrarily choose a singleton $\{X\}$ for each variety V , using the axiom of choice.

For the proof that the conditions are not vacuous, though, we are allowed to make just such an arbitrary choice. Thus first imagine that we do start with some choice of X , obtained in just that way, and let $I \subset \mathcal{O}_V(iH)$ be the sheaf generated by X .

The sheaf $R(I)$ is finite type, so the subalgebra of $\Gamma(R(\mathcal{O}_V(H)))$ generated by the particular global section generators we’ve described, whose degrees are powers of $r + 2$, must be finite type; because the full algebra of global sections is integral over the subalgebra. Our generating spaces, let us call them $X_j \subset H^0(V, \mathcal{O}_V((r + 2)^j((r + 1)iH + K)))$, because they only occur in power of $(r + 2)$ degree,

must satisfy

$$X_j^{r+2} = X_{j+1}$$

for suitably large j . For such a j take now $X = X_j$ and replace i by $i(r+2)^j$. We arrive then at a situation where our new choice of X satisfies $X_0^{r+2} = X_1$.

We can explicitly calculate X_0^{r+2} and X_1 from X . The former is the image of

$$S^{r+2}\Lambda^{r+1}(Y \otimes X) \rightarrow H^0(V, \mathcal{O}_V((r+2)(H + (r+1)iH + K_V)))$$

sending a symmetric product of terms $(y_0 \otimes x_0) \wedge \dots \wedge (y_r \otimes x_r)$ to the corresponding product of sections of the $w(y_0x_0, \dots, y_rx_r)$. The latter is the image of the map

$$\Lambda^{r+1}(Y \otimes \Lambda^{r+1}(Y \otimes X)) \rightarrow H^0(V, \mathcal{O}_V((r+1)(H + (r+1)[H + (r+1)iH + K] + K))$$

with the same target, sending an exterior product of monomials $z_\alpha(y_0x_0 \wedge \dots \wedge y_rx_r)$ to the value of w at the $z_\alpha w(y_0x_0, \dots, y_rx_r)$.

The image of the first map is contained in the image of the second, and equality of the targets is controlled by a determinantal rank formula in terms of the coefficients of the homogeneous polynomials. When it holds, the map $L_0^{r+2} \rightarrow L_1$ is an isomorphism. This map pulls back on V_1 (the Nash blowup of V_0 which is the blowup of I), to the map of the r 'th exterior powers of the differentials of V_0 pulled back to V_1 and reduced modulo torsion to that of V_1 . This is surjective, and it follows that V_0 , the blowup of the base locus of X , is an immersed image of a nonsingular variety.

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