Euler equations

Definitions in Riemannian geometry.

We begin with a possibly non-compact, orientable Riemannian manifold (without boundary) M of dimension n.

Let's begin with a review of basic Riemannian geometry. Although the smooth functions form a ring known as $C^\infty(M, \mathbb{R})$ we'll use sheaf notation in order that theorems will generalize to the analytic setting. Thus we'll consider the sheaf $\mathcal{O}_M$ of smooth functions in place of only its global section ring $C^\infty(M, \mathbb{R})$.

We will use the

$$\langle v, \omega \rangle$$

when we wish to pair together a vector field v with a one-form ω using the duality between them (as locally free sheaves over $\mathcal{O}_M$). The same function is equal to the contraction of ω along v and so

$$\langle v, \omega \rangle = i_v(\omega).$$

And if ω is closed, so $\omega = df$ it is also just equal to the (Lie) derivative $v(f)$. We'll use exactly the same angle brakcet notation

$$\langle v,\ w \rangle$$

when we pair two vector-fields using the Riemannian metric. Thus for instance in local coordinates $u_1, ..., u_n$ we have

$$\langle \frac{\partial}{\partial u_i},\ \frac{\partial}{\partial u_j} \rangle = g_{ij}$$

the i, j matrix entry of the Riemannian metric tensor in the corresponding bases. We will use subscripts throughout, and the variance of operators will not be indicated by use of superscripts in place of subscripts.

The same angle bracket notation also will be used to pair two one-forms under pairing coming from transfer of structure the isomorphism between vector fields and one-forms coming from the metric, we not only will use the same notation, but the pairings correspond.

The gradient ∇f of a function (=local section of $\mathcal{O}_M$) is to be the vector field that corresponds to the one-form df under this isomorphism. Thus for all functions f, g it is a matter of definition that

$$\langle \nabla f,\ \nabla g\rangle = \langle df,\ dg\rangle.$$

We'll write $grad(f)$ as a synonym of ∇f in formulas where there are already too many triangles and notation becomes confusing.

The natural pairing between one-forms and vector fields is related to the volume form η by the rule

$$\omega \wedge i_\delta(\eta) = \langle \delta, \omega\rangle \eta.$$

For instance if $\eta = f du_1 \wedge \ldots \wedge du_n$ and $\omega = du_1$ then

$$\begin{aligned}\omega \wedge i_\delta(\eta) &= \omega \wedge \sum_{j=1}^{n} (-1)^j f\delta(u_j) du_1 \wedge \ldots \wedge du_{j-1} \wedge du_{j+1} \wedge \ldots \wedge du_n \\ &= f\delta(u_1) du_1 \wedge \ldots \wedge du_n \\ &= \delta(u_1)\eta\end{aligned}.$$

On the other hand, the differential of $i_\delta(\eta)$ is $di_\delta(\eta)$, which is the same as the Lie derivative $\delta(\eta)$. The eigenvalue of the action of δ on η (which is a section of $\mathcal{O}_M$) is the 'volume multiplying factor' of δ, known as the divergence $div(\delta)$, and so we have

$$di_\delta(\eta) = div(\delta)\eta.$$

It follows that

1. Proposition. For any vector field δ, $div(\delta)\eta$ is an exact n form.

Remark. Note that this implies that when M is compact the divergence operator is not surjective; I think the cokernel is represented by the constant function 1 in that case.

If we replace δ by a multiple $f\delta$ with f a section of $\mathcal{O}_M$ we have

$$div(f\delta)\eta = di_{f\delta}(\eta) = d(fi_\delta(\eta))$$

just because i_δ is $\mathcal{O}_M$ linear in the subscript variable, and this then equals

$$df \wedge i_\delta(\eta) + f\ div(\delta)\eta = \langle \delta,\ df\rangle\eta + f\ div(\delta)\eta$$

Thus equivalently

$$div(f\delta) = \langle \delta,\ df\rangle + f\ div(\delta)$$

We can solve this giving a useful way of rewriting the ordinary e-valuation of δ against df – which is the value of the derivation $\delta(f)$, as

$$\begin{aligned}\delta(f) &= \langle \delta,\ df\rangle \\ &= div(f\delta) - f\ div(\delta)\end{aligned}$$

Thus,

2. Proposition. If δ is a global divergence-free vector field, then for any global compactly-supported smooth function f

$$0 = \int_M \delta(f)\eta.$$

Proof. The formula above shows that the integrand is a divergence times η, and we can apply Proposition 1. QED

Biot-Savart type rules.

The Laplace operator (more rigorously we should call it the Laplace-Beltrami operator) assigns to a section f of $\mathcal{O}_M$ the section

$$\Delta(f) = div\ \nabla f.$$

This is the volume eigenvalue of the gradient flow of f. A function f is deemed harmonic if $0 = \Delta(f)$.

Using the earlier rule we may rewrite $\langle df,\ dg\rangle$ for f, g local sections of $\mathcal{O}_M$ in two different ways, as we may write

$$df \wedge i_{\nabla g}(\eta) = \langle f,\ g\rangle = dg \wedge i_{\nabla f}(\eta).$$

The difference being zero means that we may simplify the deRham differential

$$d(fi_{\nabla g}(\eta) - gi_{\nabla f}(\eta)) = (f\Delta(g) - g\Delta(f))\eta\,.$$

Thus in particular,

3. Proposition. If g is harmonic and U is a codimension zero submanifold with boundary in M then g is an integrating factor for $\Delta(f)$ in the sense that

$$\int_U g\Delta(f)\eta = \int_{\partial U} fi_{\nabla g}(\eta) - \int_{\partial U} gi_{\nabla f}(\eta).$$

Proposition 3 leads directly to a Biot-Savart law in Euclidean space if we take U to be the complement of a sphere, M to be the one-point compactification of Euclidean space, and $g = r^{2-n}$ for r the distance from center point x of the sphere. Then $f\ i_{\nabla g}(\eta) = f\ (2-n)r^{1-n}i_{\nabla r}(\eta)$ which integrates to a number which approaches $(2-n)f(x)$ times the area of a unit sphere as the sphere shrinks to zero, the other boundary integral is negligible. Thus one may solve for $f(x)$ knowing $\Delta(f)(x)$ for all x.

Covariant differentiation of one-forms.

The covariant derivative is perhaps easiest understood in the first instance in its action on symmetric powers of the sheaf Ω_M of one-forms. For the independent direct sum of all the symmetric powers of Ω_M is contained within the sheaf of smooth functions $\mathcal{O}_{TM}$ where TM is the tangent bundle of M, just viewed as a manifold.

The metric tensor provides an extension of every vector field on M to a vector field on TM whose restriction to the zero section M is this given vector field. If we write δ for the derivation of $\mathcal{O}_M$ associated to a vector field on M, we then can write ∇_δ for the corresponding derivation of $\mathcal{O}_{TM}$.

If the vector field is only defined on an open subset U (as we allow when speaking of the sheaf of vector fields), then it describes a derivation of $\mathcal{O}_U$ and extends to a derivation on $\mathcal{O}_V$ where $V \subset TM$ is the inverse image of U under the tangent bundle projection.

Thus ∇_δ acts on functions on the tangent bundle in the ordinary way, and if ω is a one-form on M we can interpret it as being a real-valued smooth function $TM \to \mathbb{R}$ which happens to restrict to a linear transformation on the linear fiber spaces.

Note that ordinary functions, sections of $\mathcal{O}_M$, can be viewed as sections of $\mathcal{O}_{TM}$ which just happen to be constant on the tangent bundle fibers. Then the action of ∇_δ on such functions is just the same as the action of δ.

This observation can be understood as explaining the 'rule of a connection,' namely that if f is a function and ω is a one-form

$$\nabla_\delta(f\omega) = \nabla_\delta(f)\omega + f\nabla_\delta(\omega)$$

simply because f and ω are both functions with domain TM and we have applied a derivation to their product. But then since $\nabla_\delta(f) = \delta(f)$ we can rewrite this

$$= \delta(f)\omega + f\nabla_\delta(\omega).$$

The defining rule of an affine connection that ∇ is $\mathcal{O}_M$ linear in its subscript variable, and satisfies Leibniz rule in this sense. Note that this rule can be used to calculate ∇_δ on any symmetric function of one-forms, and these are dense in the function space on TM in various settings.

It follows that the action of ∇_δ in local coordinates $u_1, ..., u_n$ is described by a matrix ω_j^k of one-forms[1] so that

$$\nabla_\delta(du_j) = \sum_k i_\delta(\omega_j^k)dx_k.$$

Writing the one-forms in the local basis $du_1, ..., du_n$ one usually now uses a negative sign

$$\omega_j^k = -\sum_i \Gamma_{i,j}^k dx_i.$$

The reason for the negative sign is that we can extend the action of ∇_δ to act on vector fields (which in some sense would have been a more familiar starting place), and the fact that the pairing between one-forms and vector fields is natural means it is invariant under this flow on the tangent bundle. Thus

$$\nabla_\delta\langle du_i, \frac{\partial}{\partial u_j}\rangle = \langle\nabla_\delta(du_i), \frac{\partial}{\partial u_j}\rangle + \langle du_i, \nabla_\delta(\frac{\partial}{\partial u_j})\rangle$$

and the expression on the left side being the derivative of 0 or 1 is equal to zero.

[1] We'll use a superscript for k only because this is how it later is used in Γ_{ij}^k in standard notation, but the choice of using a letter as a superscript or subscript has no significance in our treatment, and all systems of coordinates are arbitrary (not required to be 'holonomic')

Such considerations determine exactly what the functions $\Gamma^k_{i,j}$ must be, as we may deduce from

$$
\begin{aligned}
\frac{\partial}{\partial k} g_{ij} &= \frac{\partial}{\partial k}\langle du_i,\ du_j\rangle \\
&= \langle \nabla_{\frac{\partial}{\partial u_k}} du_i, du_j\rangle \\
&+ \langle du_i,\ \nabla_{\frac{\partial}{\partial u_k}} du_j\rangle \\
&= -\sum_s \Gamma^s_{ki}\langle du_s, du_j\rangle - \sum_s \Gamma^s_{kj}\langle du_s, du_i\rangle \\
&= -\sum_s \Gamma^s_{ki} g_{sj} - \sum_s \Gamma^s_{kj} g_{si}
\end{aligned}
.
$$

From symmetry of both g and Γ in its subscripts (discussed elsewhere) one obtains Γ^k_{ij} as the solutions of the linear equations [2]

$$2\sum_l g_{lr}\Gamma^l_{jk} = \frac{\partial}{\partial u_k} g_{rj} + \frac{\partial}{\partial u_j} g_{rk} - \frac{\partial}{\partial u_r} g_{jk}.$$

which can be obtained as a linear combination of three cyclic rotations of the indices in the earlier equation.

One consequence[3] is that if we write the volume form locally as $hdu_1 \wedge ... \wedge du_n$ then just as the divergence of $\frac{\partial}{\partial u_i}$ is expressed as the action by multiplying this form by the eigenfunction

$$\frac{\partial}{\partial u_i} log\ h,$$

when we calculate the divergence of an arbitrary local vector field $\delta = \sum c_i \frac{\partial}{\partial u_i}$ we obtain

$$
\begin{aligned}
div(\delta) &= \sum_i c_i \frac{\partial}{\partial u_i}\ log\ h \\
&= \sum_{i,j} c_i \Gamma^i_{ij} \\
&= trace \nabla(\delta)
\end{aligned}
.
$$

[2] see Wikipedia

[3] see online notes by Min Ru at Houston University

Here ∇ is the action on vector fields (conjugate to the action on one-forms by the isomorphism between them coming from the metric), and $\nabla(\delta)$ is the $\mathcal{O}_M$ linear map sending a vector field v to the vector field $\nabla_v(\delta)$.

Geodesic flow.

The geodesic flow is the vector field, let us call it ϕ, on TM coming from directional derivative. It assigns to a function f the function df viewed as a function whose domain is the tangent bundle.

In turn, to a closed one-form $\omega = df$ on M viewed as a particular type of function on TM, it assigns $\sum \nabla_{\frac{\partial}{\partial u_i}}(f) \otimes du_i$ where the tensor is a symmetric tensor (but if ω is not closed it will not in general be symmetric).

In higher degrees the action is determined by Leibniz rule, at least on the symmetric algebra sheaf.

Thus, the geodesic flow is a derivation whose restriction to the symmetric algebra sheaf $\mathcal{O}_M \oplus \Omega_M \oplus S^2(\Omega_M) \oplus ...$ increases degrees by one.

The image in M of an orbit of the geodesic flow in TM is a geodesic in M.

The Euler equations in terms of E and F.

The 'curl' of a vector field corresponds under the isomorphism between vector fields and forms to the deRham differential

$$\Lambda^1\Omega_M \to \Lambda^2\Omega_M.$$

In many situations it makes sense to choose a one-sided inverse of the Laplacian, a function τ with domain of definition the image of div, such that $identity = \Delta \circ \tau$. This follows in some situations from the Biot Savart calculation (subsequent to Proposition 3) above. Also, in analytic situations or algebraic situations where it makes sense to consider a single origin and a radial function r, harmonic functions are a complement to multiples of r^2 and again the Laplacian is surjective, this time with one-sided inverse τ so that

$$identity = \Delta \circ \tau.$$

In such analytic (or formal or algebraic) cases one can take τ to be r^2 times the inverse of $grad \circ div \circ r^2$.

When M happens to be compact, the divergence operator cannot be surjective (see the remark following Proposition 1). Assume now that we have found such a $\tau : \mathcal{O}_M \to \mathcal{O}_M$ satisfying the rule that

$$\Delta \circ \tau - identity \quad \text{is zero on the image of } div \quad (1)$$

It is immediate that the two functions

$$E = grad \circ \tau \circ div$$

$$F = 1 - E$$

satisfy the three rules

$$\left\{ \begin{array}{l} E + F = 1 \\ 0 = div \circ F \\ 0 = curl \circ E \end{array} \right. \qquad (2)$$

Conversely, from these equations (2) about E and F and the extra assumption $H^1(M, \mathbb{R}) = 0$ one can deduce from $curl \circ E = 0$ the rule $E = grad \circ \phi$ for some operator ϕ from vector fields to functions. Since div is surjective one can write $\phi = \tau \circ div$. Then $E = grad \circ \tau \circ div$.

Finally

$$\begin{aligned} div = div \circ identity &= div \circ (E + F) \\ &= div \circ E \end{aligned}$$

since $0 = div \circ F$, and from this

$$div \circ grad \circ \tau \circ div = div \circ E = div$$

showing that τ is a one-sided inverse of $\Delta = div \circ grad$ when restricted to the image of div as required.

This shows

4. Proposition. There is a bijection between pairs E, F of real linear operators satisfying (2) and real lienar operators τ satisfying (1).

Assume now that E, F are as in (2). Let us define the corresponding Euler equations for a time zero divergence free vector field v to be the condition that the once extended in time to a time-dependent vector field, v satisfies

$$\frac{d}{dt}v = -F\nabla_v(v). \qquad (3)$$

I have stated the equation as a type of definition only because I've included the role of the operator F in the formula, rather than in each case to remark that one or another maximum principle or boundary condition implicitly specifies the correct 'divergence free part' of self-directional derivative. Note very clearly that this is then just an ordinary differential equation.

Conservation laws.

If w is any divergence free vector field

$$\int_M \langle F\nabla_v(v), w\rangle\eta = \int_M \langle \nabla_v(v) - E\nabla_v(v), w\rangle\eta.$$

If M has trivial first deRham cohomology we may write $E\nabla_v(v)$ as a gradient

$$E\nabla_v(v) = \nabla(p)$$

and then

$$\begin{aligned}\int_M \langle E\nabla_v(v), w\rangle\eta \\ &= \int_M \langle \nabla p, w\rangle\eta \\ &= \int_M \langle dp, w\rangle\eta = 0\end{aligned}$$

by Proposition 2, since w is divergence free.

This shows

6. Proposition When v satisfies the Euler equation, then for any w compactly-supported and divergence free, if $0 = H^1(M, \mathbb{R})$ then

$$\int_M \langle \frac{d}{dt}v, w\rangle\eta = \int_M -\langle \nabla_v(v), w\rangle\eta.$$

Note that what this means is that on average (after integrating against η) the flow on vector fields which occurs in the Euler equation cannot be distinguished from its divergence free part (the result of applying F) Finally

7. Proposition If v is divergence free and w is a compactly supported vector-field then

$$0 = \int_M \langle \nabla_v(w), w\rangle\eta.$$

Proof. This is

$$\frac{1}{2}\int_M \nabla_v \langle w, w\rangle \eta = \frac{1}{2}\int_M v(\langle w, w\rangle)\eta$$

which is zero by Proposition 2.

The combination of Proposition 6 and Proposition 7 with $w = v$ shows that whenever $\int_M \langle \frac{d}{dt}v, r\rangle \eta = \frac{d}{dt}\int_M \langle v, v\rangle \eta$, that is whenever we may commute the time derivative with the integral, the 'energy' $\int_M \langle v, v\rangle \eta$ is constant as a function of time.

Proposition 7 is in some sense a traceless condition, it says that on average the action of covariant differentiation along a divergence free vector field has no 'radial' component. It is stronger than a traceless condition since it implies vanishing of all eigenvalues or eigenfunctions rather than their sum, but weaker since it requires averaging over M.

By contrast, another traceless condition applies to $\nabla_v(w)$ for each fixed divergence-free w, it is traceless on the nose as a linear function of v, being true point-by-point rather than only after being integrated against a divergence-free vector feld; yet not implying separate eigenvalues are zero.

Power series methods (formal existence).

Now let's discuss a little the choice of τ or equivalently of the decomposition $1 = E + F$.

We may iterate the Euler equation (3) to obtain a recursive equation for all the higher time derivatives of v

$$\frac{d}{dt}^i v = -\sum_{\beta=0}^{\alpha-1} \binom{\alpha-1}{\beta} F\nabla_{\frac{d}{dt}^\beta v}(\frac{d}{dt}^{\alpha-1-\beta} v) \quad (4)$$

This may be unwound into a formal power series

$$v(t) = \sum_{i=0}^{\infty} \frac{a_i}{i!} t^i$$

where the coefficient a_i is the right side of (4) substituted recursively to remove all occurrences of $\frac{d}{dt}^u$ for all numbers u besides zero.

Also, if one wishes, one may write E, F in terms of the single operator τ, and if one has chosen an explicit formula for τ, either the Biot-Savart formula described above, or the formula $\tau = r^2(grad \circ div \circ r^2)^{-1}$ of formal power series, the a_i become explicit expressions either involving the integrals in the Biot Savart law, or involving the linear algebra finite-dimensional matrix inverse degree-by-degree of the action of $\Delta \circ r^2$ on a graded polynomial algebra.

The uniqueness of τ is only up to adding a function from the image of div to the space of harmonic functions; and if we independently adjust every occurrence of τ in such a formula, it gives another formal solution to the Euler equation. These solutions are all very independent of each other. The sense in using one fixed function τ in every occurrence is that if one is working with compactly supported vector fields with a boundary condition satisfied by no nonzero harmonic functions, then τ is uniquely specified.

In the case of Euclidean space and compactly supported vector fields, on can uniquely write

$$\tau = r^2(grad \circ div \circ r^2)^{-1} + h$$

where h is a function from the image of *div* to the harmonic functions, chosen so that the sum which is τ satisfies the boundary condition of being compactly supported. Then each time τ is applied in the formula (4) to a function f $h(f)$ will agree with $-r^2(grad \circ div \circ r^2)^{-1}(f)$ except on a compact set, and by the maximum principal will be bounded by the maximum value of this function on any particular fixed larger compact set. Then the sum can have no value larger than twice this maximum.

The same consideration applies more generally to any choice of τ which may then be made without regard to the boundary conditions, and adjusted by the addition of an operator from the image of *div* to harmonic functions, and the adjustment does not significantly affect the convergence bound of each separate occurrence of τ in (4).

Evidence of non-existence of global solutions in the generic three-dimensional case

It is very very likely that there are counterexamples to existence in $\mathbb{R}^3$ of all-time solutions for the case of analytic rapidly decreasing initial conditions.

When we do correctly iterate the derivation $v \mapsto -F\nabla_v(v)$, in the first instance just removing F altogether, we have self-directional derivative.

Then the second derivative of v is $-\nabla_v(-\nabla_v(v)) - \nabla_{\nabla_v(v)}(v)$ and so-on.

It is best to write the operation $(w, v) \mapsto \nabla_v(w)$ as

$$w * v = \nabla_v(w)$$

Then the failure of associativity is in the formula

$$(a * b) * c - a * (b * c) = hess(a)(b, c)$$

where $hess(a)$ is the vector whose entries are the $Hessians$ of the entries of a.

That is,

$$\nabla_c(\nabla_b(a)) = \nabla_{\nabla_c(b)}(a) + hess(a)(b, c).$$

Now, we are *not* allowed to ignore all higher than linear terms (quadratic etc) exactly because of this Hessian term.

If we could, then we could write each term which looks like some associative word in the single letter v and the operation *, in the form

$$v * (v * (v * (...v)))$$

The point is that the coefficient is now something like $(i + 1)!$ if i is the depth of the word.

This means that if we look at one point, before we've adjusted by adding in the gradient of the harmonic function or the Hessian terms anywhere, we have something which is a power series in t along with radius of convergence at most 1, generically. Barring any unseen relations here.

Now, here is where the case of $\mathbb{R}^2$ is strikingly different than $\mathbb{R}^3$.

For 'linear' vector fields (where the coefficients are just linear functions of $u_1, ..., u_n$ the coordinates), we have the following:

divergence -free linear vector fields $\leftrightarrow$ traceless matrices

grad's of homogeneous harmonics of degree 2 $\leftrightarrow$ traceless symmetric matrices

Now, if all we do is start with a divergence free vector field, look at its linear part, all we are doing is that if we view v as the operation sending x to $v * x$, then the $*$ operation is just matrix multiplication

And the map we are iterating is just alternating

squaring

subtracting the $1/n$ times the trace of the matrix, times the identity.

Now, in case $n = 2$, this operation

$$M \mapsto M^2 - (1/2)trace(M^2)I$$

is finite order, in the sense that starting with any traceless matrix, in one iteration it becomes symmetric (it is zero if you start with a nilpotent matrix). And in the next step it becomes zero..

Then it does not much matter what gradient of a harmonic function you add, that is, what symmetric matrix you add.

That is to say, you do not need to be at all careful to get convergence, there is no interaction between the gradient of the harmonic function and the underlying self-directional derivative.

The same calculation with $n = 3$ shows that there is now a really stringent demand on the linear part of the gradient of the harmonic function.

If we start with a compactly supported smooth vector field with lots of linear regions in different places, do we really expect that the error term in the failure of associativity will combine exactly with the symmetric matrix, to equal the symmetric part of M?

And if this does not happen, then what we are seeing is terms added together indexed by a tree, and this is a tree whose branching starting at any point goes by more than the factorial function.

Now, the thinking is mixing up analytic and smooth arguments. It is possible for the radius of convergence to be finite and for there to be no singlarities, they could be non-real singularities, or the Taylor series could have zero radius of convergence and be unrelated to the known local solutions.

But it is likely that the finite radius of convergence matches the known local solutions, and that the failure of finiteness of the radius of convergence really means that things tend to infinity.

And that for general rapidly vanishing analytic solutions, it would be too much of a coincidence to expect the gradients of the harmonic functions plus the error term from associativity to always cancel, and it is likely that when they do not you get a radius of convergence near 1 which represents actual failure of the solution to extend even as a real smooth solution.

Uniqueness of solutions of Euler's equations.

Here is a clever proof of uniqueness which I found in Majda's book (I do not know the full origin of it). If there were two solutions with the same initial condition, call these $v, v+\epsilon$, in the compactly supported case, say, then we can use Proposition 5 which says that F does not matter, and proposition 6, vanishing of eigenfunctions.

We write

$$\frac{d}{dt}\epsilon = -F(\nabla_{v+\epsilon}(v+\epsilon) - \nabla_v(v))$$

and then by divergence-freeness of ϵ as in Proposition 5 we may remove the occrrences of F as long as we pair with a divergence free field, multiply by η and integrate over M, giving

$$\int_M \langle \frac{d}{dt}\epsilon, \epsilon\rangle \eta = \int_M \langle -\nabla_{v+\epsilon}(v+\epsilon) + \nabla_v(v), \epsilon\rangle \eta.$$

The clever step is to rewrite the first argument

$$\nabla_{v+\epsilon}(v+\epsilon) - \nabla_v(v) = \nabla_{v+\epsilon}(\epsilon) + \nabla_\epsilon(v).$$

Then

$$\langle \frac{d}{dt}\epsilon, \epsilon\rangle = \langle -\nabla_{v+\epsilon}(v), \epsilon\rangle - \langle \nabla_\epsilon(v), \epsilon\rangle$$

and Proposition 6 says that the first term integrates to zero, giving

$$\int_M \frac{d}{dt}\langle \epsilon, \epsilon\rangle = -2\int_M \langle \nabla_\epsilon \; v, \epsilon\rangle.$$

which is bounded above by a constant C times $\langle \epsilon, \epsilon\rangle$. In coordinates $u_1, ..., u_n$ the constant C can be bounded in terms of the maximum magnitude on the compact support set of all the $i_{\frac{\partial}{\partial u_\alpha}}\omega_{\beta,\gamma}$ and on the coefficients of v and their partial derivatives in the basis $\frac{\partial}{\partial u_\alpha}$. Thus

$$|\frac{d}{dt}log(\langle \epsilon, \epsilon\rangle)| \leq C$$

One can finish the uniqueness proof with an very elementary proposition about real functions

8. Proposition. Suppose $f(t)$ is differentiable real-valued function and there is a number C such that $|\frac{d}{dt}logf(t)| \leq C$ at all points t where $f(t) \neq 0$. Then the closed zero set $\{t : f(t) = 0\}$ is also open, so it is either empty or the whole of $\mathbb{R}$.

Proof. We know the set of t such that $f(t) = 0$ is a closed set. If it were not open there would be points t with $f(t) \neq 0$ tending to a point a with $f(a) = 0$. Then $logf(t)$ would tend to infinity as $t \to a$ and its derivative would be unbounded in magnitude by the mean value theorem. That then finishes the uniqueness proof in the textbook – we've introduced the proposition above as a replacement of an estimating technique of Gronwall.

The main lessons so far seem to be that it is not difficult to give a sort of formal analytic expression for the soltion that is, for instance actually a formal series if the initial time zero vector field is analytic; yet the convergence issue is complicated.

The complication about the choice of harmonic part in each step of the recursion is nearly trivial, it can be removed just by integrating against a divergence free vector field. But when one looks at this, as the steps of the recursion comine, there is not yet known a clear way to remove the second and higher order effects which would affect the self-directional derivatives if the harmonic error were not actually subtracted away point by point.

The same is true, more powerfully, for the projection onto the divergence free part. That the projection acts as the identity if one is only going to integrate against a divergence free vector field, but this need not be true of the *second order* effect if the unprojected vector field were used later as one of the arguments in self-directional derivative. I am not saying that this issue is impossible, only that it is very complicated.

Geodesic flow and directional derivative.

This section attempts to clarify the relation between geodesic flow and directional derivative. There is not yet any particular meaningful observation related to the self-directional derivative which occurs in Euler's equations.

Let's return relate the geodesic flow on the tangent bundle TM with directional derivatives of vector fields.

Let us work analytically for reasons of simplicity. As I've mentioned elsewhere, the cotangent sheaf of TM is spanned by $\mathcal{O}_M$ and Ω_M and therefore the pushforward to M is

$$(\mathcal{O}_M \oplus \Omega_M \oplus S^2(\Omega_M)...)d'\mathcal{O}_M + (\mathcal{O}_M \oplus \Omega_M \oplus S^2(\Omega_M)...)d'\Omega_M$$

where we use d' to refer to the deRham differential on TM. The sum is not direct, however the summand

$$\Omega_M d'\mathcal{O}_M + \mathcal{O}_M d'\Omega_M$$

is unaffected as a set by the removal of the coefficient sheaf $\mathcal{O}_M$ in the second summand, and then becomes a direct sum. That whole summand (which is now a direct sum) is a copy of one forms on TM which restrict to zero as one forms on the zero section M, then restricted to zero in the sheaf sense. It is the sheaf of first principal parts of Ω_M and in this way can be interpreted as a residue.

Contraction along geodesic flow is a linear map from this to $\mathcal{O}_{TM}$ which gives a map over $\mathcal{O}_M$ to the pushforward of $\mathcal{O}_T M$ which is

$$\mathcal{O}_M \oplus \Omega_M \oplus S^2(\Omega_M) \oplus ...$$

The linear map sends the whole of the principal parts summad to $S^2\Omega_M$. On $\Omega_M d'\mathcal{O}_M \cong \Omega_M^{\otimes 2}$ the map is merely symmetrization, this corresponds to the fact that $\mathcal{O}_M d'\mathcal{O}_M$ is a summand of the whole first factor in the main decmposition, and this is just the pushforward of the pullback of one-forms (the 'horizontal' one forms) upon which contraction along geodesic flow is the identity on closed forms. That is to say, contraction along the geodesic flow sends $d'f$ to df which does nothing to a function f constant along tangent bundle fibers except to replace the symbol d' for the tangent bundle deRham operator with that for the manifold M itself.

Since the principal parts summand generates Ω_{TM} the whole information about the action of geodesic flow is then determined by the map $\mathcal{O}_M d'\Omega_M \to S^2\Omega_M$. And this is determined by the map $d'\Omega_M \to S^2\Omega_M$ sending $d'\omega$ to $\nabla(\omega)$, for each particular one-form ω on M, interpreted not as a linear map from the tangent sheaf to the cotangent sheaf, but as an element of the tensor square of Ω_M which happens to land in the symmetric tensors.

Let's denote by ϕ the geodesic flow vector field on TM. The restriction of the contracting map i_ϕ to $\mathcal{O}_M d'\Omega_M$ is $\mathcal{O}_M$ linear, and the further restriction to $d'\Omega_M$, is the connection ∇.

It can give insight if we consider the restriction not to $d'\Omega_M$ but to the larger $\mathcal{O}_M d'\Omega_M$. This is the linear map sending $\sum g_i d'\omega_i$ to $\sum g_i \nabla(\omega_i)$ for one-forms ω_i on M.

There is the relation between d and d' coming from the fact that df is a function with domain TM and d' satisfies Leibniz rule

$$d'(hdf) = hd'(df) + (df)d'h.$$

Or more generally for ω a one-form on M

$$d'(h\omega) = hd'(\omega) + \omega d'h.$$

From this

$$\begin{aligned}\nabla(h\omega) &= i_\phi(d'(h\omega)) \\ &= i\phi(hd'(\omega) + \omega d'h) \\ &= hi_\phi(d'(\omega)) + \omega i_\phi(d'h) \\ &= h\nabla(\omega) + \omega dh\end{aligned}.$$

The product ωdh here is the symmetric tensor $\frac{1}{2}(\omega \otimes dh + dh \otimes \omega)$.

Thus it is consistent and correct to interpret the linear map which is the restriction of the contraction i_ϕ along geodesic flow ϕ as an extension of ∇ to an $\mathcal{O}M$ linear map $\mathcal{O}_M d'\Omega_M \to S^2\Omega_M$.

Covariant derivatives and directional derivatives.

The relation between the covariant derivatives of tangent and cotangent bundles ought to eventually relate $\nabla(\omega)$ with self-directional derivatives of vector fields that occur in the Euler equations. Let us attempt to understand this some more. In local coordinates $u_1, ..., u_n$ we may start with the adjunction for each fixed vector field δ,

$$\langle \nabla_\delta(du_p), \frac{\partial}{\partial u_q} \rangle = -\langle du_p, \nabla_\delta(\frac{\partial}{\partial u_q}) \rangle$$

which holds since the $\langle du_p, \frac{\partial}{\partial u_q} \rangle$ are sent to zero under ∇_δ.

Writing in the same coordinates $u_1, ..., u_n$ the equivalent tensor (we'll use an equals sign as an abuse of notation)

$$\nabla(du_p) = \sum_i \nabla_{\frac{\partial}{\partial u_i}}(du_p) \otimes du_i$$

we have for indices α, β that

$$\langle \nabla(du_p), \frac{\partial}{\partial u_\alpha} \otimes \frac{\partial}{\partial u_\beta} \rangle$$

$$= \sum_i \langle \nabla_{\frac{\partial}{\partial u_i}}(du_p) \otimes du_i, \frac{\partial}{\partial u_\alpha} \otimes \frac{\partial}{\partial u_\beta} \rangle$$

$$= \sum_i \langle \nabla_{\frac{\partial}{\partial u_i}}(du_p), \frac{\partial}{\partial u_\alpha} \rangle \langle du_i, \frac{\partial}{\partial \beta} \rangle$$

This is zero unless $\beta = i$ in which case it is the single term

$$\langle \nabla_{\frac{\partial}{\partial u_\beta}}(du_p), \frac{\partial}{\partial u_\alpha} \rangle.$$

Now applying the adjunction this is

$$-\langle du_p, \nabla_{\frac{\partial}{\partial \beta}}(\frac{\partial}{\partial \alpha}) \rangle.$$

If one now calculates for $v = \sum_\alpha a_\alpha \frac{\partial}{\partial u_\alpha}$, $w = \sum_\beta b_\beta \frac{\partial}{\partial u_\beta}$

$$\langle \nabla(du_p), v \otimes w \rangle$$

is not quite $\mathcal{O}_M$ bilinear in v, w, It evaluates to

$$-\sum_\beta b_\beta \langle du_p, \nabla_w(\frac{\partial}{\partial u_\beta})\rangle$$

and so

$$\langle \nabla(du_p), v \otimes w\rangle = -\langle du_p, \nabla_w(v) - \sum_i w(a_i)\frac{\partial}{\partial u_j}\rangle.$$

The second term would be the value of $\nabla_w(v)$ if $u_1, .., u_n$ were Euclidean coordinates.

Thus we have seen that when we apply the contraction operator i_ϕ with ϕ the geodesic flow to a function f on M viewed as a function on TM constant on tangent space fibers, we get the answer of df, and in turn if we apply i_ϕ it to a one-form of the type $d'\omega$ with ω a one-form on M we get $\nabla(\omega)$, the action of covariant differentiation on the one-form ω.

In the case $\omega = du_p$ is a closed form, the result $\nabla(du_p)$ is adjoint to a bilinear expression which might be interpreted to be the the non-Euclidean part of directional derivative.

The action of i_ϕ is $\mathcal{O}_{TM}$ linear and therefore the action is $\mathcal{O}_M$ linear on sums of expressions of the type $ad'\omega$ with a functions on M and ω one-forms on M. But this is linearity with respect to the action by multiplications which affect a, not b, and does not commute with the isomorphism of Ω_M to its image $d'\Omega_M$ viewed as one-forms on TM (with Ω_M viewed as functions).

The restriction of i_ϕ to $d'\Omega_M$ while not linear in this sense, is equal to the action ∇ of covariant differentiation on one-forms, sending closed one-forms to symmetric tensors in $\Omega_M \otimes \Omega_M$.

The relation with Burger flow

If v is a vector field on M, then the time-dependent vector-field, if it exists, determined by

$$\frac{dv}{dt} = -\nabla_v(v)$$

can represent the effect on tangent vectors (which are implicitly measured relative to the zero-section, that is, implicitly *stationary coordinates*), of a flow which is forceless. The Euler equations

$$\frac{dv}{dt} = -F \circ \nabla_v(v) = -\nabla_v(v) + E\nabla_v(v)$$

$$= -\nabla_v(v) + grad \circ \tau \circ div(\nabla_v(v))$$

can be most easily understood if we return to thinking of the divergence of a vector field v as the trace of the linear operator $\nabla(v)$. From Leibniz rule,

$$div\nabla_v(v) = trace\nabla_{\_}\nabla_v(v)$$

$$= \nabla_v trace(\nabla_{\_}v) + trace\nabla_{\nabla_{\_}v}v$$

As v is divergence free for all time when it solves the Euler equations (once the initial time-zero vector field is divergence free) the first term is zero and we have

$$= trace\nabla_{\nabla_{\_}v}v$$

$$= trace(\nabla v)^2.$$

Thus the Euler equations become

$$\frac{dv}{dt} = -\nabla_v(v) + grad \circ \tau \; trace(\nabla v)^2.$$

When the Biot-Savart calculation converges, the second term is exactly the constant $\frac{1}{(n-2)Area(S^{n-1})}$ times what would be the electrostatic attraction/repulsion from a charge distribution of density function $trace(\nabla v)^2$. For example in three-dimensional space, it would be the integral of the inverse-squared distance to the charge, times this constant.

Thus, one way of intuitively understanding the Euler equations is to think that the departure from a purely inertial flow, caused by the pressure in an incompressible fluid, follows exactly the same equations as if the force were caused by an electrostatic charge distribution of magnitude given by $trace(\nabla v)^2$. [4]

[4] It is curious to wonder whether the reverse is also true, whether the equations of electrostatics can be interpreted by visualizing an incompressible fluid, and whether there is any useful relativistic formulation, whereby what are 'stationary coordinates' for one viewer are 'moving coordinates' for the other, such that the notion of an incompressible fluid, versus an inverse-square law force field, are interchanged between one viewer and the other.