I. Considerations of naturality...

An elliptic curve is a branched double cover of a Riemann sphere in more than one way. The underlying issue of naturality has its home in symmetry of torsors for the finite Klein four-group. The Jacobian of any elliptic curve contains a copy of the Klein four-group, but different elliptic curves have different Jacobians, and it is not right to say any elliptic curve is a representation of *the* Klein four-group.

This section can be subsumed into Galois theory of fields, or also into the theory of braid groups and mapping class groups, or into the theory of covering spaces of manifolds, or, really, into the theory of the finite group S_4 with its unique four-element subgroup. But it nice to start with elliptic curves without making any complicated definition of what they are, just assuming we know them as an axiomatic starting place like the plane in Euclid's theory.

I.1. ...for K_4 torsors.

When we consider the natural permutation representation on a four element set S, there is for each group element g the 'twisted' representation ${}^gS$, defined by the operation $\cdot^g$ so that

$$h \cdot^g s = g^{-1}hg \cdot s.$$

The restriction of the representation to the Klein four-group maps the 24 different twisted representations to six representations of the Klein four-group.

I.2. ... for covering spaces.

Now we can think of what this means on the level of covering spaces. Any four-sheeted cover of, let us say, a complex manifold M has a natural associated six sheeted cover $\widetilde{M} \to M$, which may be disconnected even if M is connected, and with Galois group S_4/K_4.

I.3. ... for sphere bundles

Now we can think in turn what this means for Riemann sphere bundles. The holomorphic fiber bundles with fiber a Riemann sphere with four points deleted are classified by complex manifolds M together with a S_4/K_4-equivariant period map $\widetilde{M} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Two such equivariant maps describe isomorphic bundles over M if and only if the period maps agree after composing with a translation of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ belonging to S_4/K_4, in other words a holomorphic automorphism of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. The fundamental group of $P^1 \setminus \{0, 1, \infty\}$ is the pure braid group on four strands on $\mathbb{P}^1$ and when M is connected equivariance under S_4/K_4 extends the induced map on fundamental groups to a map from the fundamental group of M to the full braid group.

For simplicity let's look at the case when M is connected. The connected components of $\widetilde{M}$ are isomorphic to each other, and each connected component is a copy of the lowest cover where the period map becomes well-defined. If M is a connected curve, this is the 'Riemann surface' of the multivalued period map on M, let us call it λ, and there is a Galois field extension $\mathbb{C}(M) \subset \mathbb{C}(M, \lambda)$ whose Galois group is the stabilizer of one connected component of $\widetilde{M}$.

That is to say,

1. Remark. The period map underlying a fiber bundle with fibers Riemann spheres with four points deleted needn't be single-valued; it can be multivalued, defining a covering space of the base manifold M of degree at most six.

I.4. ...for bundles of pointed elliptic curves.

Now let's think of what this means for actual bundles of elliptic curves with basepoint (i.e., bundles with a section). Given an elliptic curve J with chosen point p, the subgroup $2H_1(J) \subset H_1(J) \cong \pi_1(J,p)$ defines a natural four sheeted regular cover $\widetilde{J} \to J$. The inverse image of p is a four-element subset of $\widetilde{J}$ which is a torsor for unique Klein four subgroup of any of the four group structures of $\widetilde{J}$ which correspond to a choice of lift of p.

There is a unique involution of $\widetilde{J}$ with this four element subset as its fixed point set, and the quotient modulo the involution is a Riemann sphere with a marked set of four (indistinguishable) points.

Applying what we've already said about Riemann sphere bundles, we can associate to any fiber bundle of elliptic curves with a (chosen) section a Galois cover (which is possibly disconnected even if M is connected) $\widetilde{M} \to M$ of degree six and a S_4/K_4 equivariant period map $\widetilde{M} \to \mathbb{P}^1 \setminus \{0,1,\infty\}$, uniquely determined up to the six translations (=automorphisms) of $\mathbb{P}^1 \setminus \{0,1,\infty\}$.

The corresending four-sheeted cover of $\widetilde{M}$ now has a globally-defined free K_4 action. The twists of the K_4 torsor are just translations and all have become isomorphic on $\widetilde{M}$. Under the covering maps $\widetilde{J} \to J$ the four points in the torsor become identified with one basepoint in each elliptic curve fiber J and we recover the section of the elliptic curve bundle.

I.5. ...for bundles of elliptic curves.

Finally for holomorphic bundles of elliptic curves which may not have a section, and for which we have chosen no section, we must specify a complex manifold M and a bundle J of pointed elliptic curves, and an element of $H^1(M, J)$ to choose a torsor. Thus

2. Theorem. A holomorphic bundle of elliptic curves (without chosen basepoint in each) is determined by choosing a complex manifold M, a four-sheeted cover of M, an S_4/K_4-equivariant period map $\widetilde{M} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$, where $\widetilde{M}$ is the corresponding six-sheeted cover, and finally an element of $H^1(M, J)$ where J is the corresponding bundle of pointed elliptic curves. Two S_4/K_4 equivariant period maps with cohomology class determine isomorphic elliptic curve bundles if and only if the agree after an automorphism of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

3. Corollary. In a bundle of elliptic curves which is connected, compact and projective all elliptic curve fibers are isomorphic to each other.

Here we speak literally; in its common usage the term ‘elliptic fibration’ is allowed to refer to a more general situation where not all fibers are elliptic curves. In fact, it is the truth of the corollary which has led to abandoning the use of the term ‘elliptic fibration’ in its literal sense to mean a bundle of elliptic curves, as such a thing in the compact projective case is merely a torsor of a trivial bundle.

To prove the corollary, just observe that once M and therefore $\widetilde{M}$ are compact and projective, the holomorphic period map $\widetilde{M} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ is proper and has compact image. The compact analytic subsets of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ however are discrete.

II. Isogenies

Once an elliptic curve $E \to S$ is a doubly branched cover of a Riemann sphere S, then for each choice of a pair of the four branch points there is an associated cover $S' \to S$ branched only at those two points, which is another Riemann sphere. The normalized pullback $E' \to S'$ is another elliptic curve branched over a Riemann sphere. The induced map $E' \to E$ is unbranched. Thus the isogeny $E' \to E$ covers the branched cover $S' \to S$.

The map $E' \to S'$ is now unbranched at the points of E' which map to the two critical points of $S' \to S$.

The isogeny underlies a reduction step in the classical theory of elliptic integrals. The one-form

$$\frac{dz}{\sqrt{z(z-1)(z-\lambda)}}$$

whose Riemann surface is an elliptic curve branched at $0, 1, \lambda, \infty$ is transformed by the substitution $z = t^2$ to the one-form

$$\frac{2dt}{\sqrt{(t^2-1)(t^2-\lambda)}}.$$

whose Riemann surface is again an elliptic curve, now branched at $1, -1, \pm\sqrt{\lambda}$ and unbranched at $0, \infty$.

III. Construction of elliptic curves

Let s be a global section of $T_{\mathbb{P}^1}^{\otimes 2}$. Then $s(\mathbb{P}^1)$ meets the zero section $\mathbb{P}^1$ at a divisor of degree four. The inverse image of $s(\mathbb{P}^1)$ under tensor square

$$T_{\mathbb{P}^1} \to T_{\mathbb{P}^1}^{\otimes 2}$$

is a 2-section of $T_{\mathbb{P}^1}$ which also meets the zero-section at four points. If the four points are distinct, the pullback of the 2-section of the tangent bundle of $\mathbb{P}^1$ to the 2-section viewed as a double cover, splits into a pair of mutually negative 1-sections without zeroes, showing that the 2-section itself has trivial tangent bundle.

While the operation of pulling back is undefined at the branching points, four undefined points in each of the two components amount to 'removable singularities.'

IV. Compactification.

IV.1 Introduction.

This chapter will be a special case of exercise 17 on page 25 of the Park City notes on surfaces by Miles Reid: the case $a = 1$ and $\alpha = 4$. We continue investigating elliptic curves, now knowing that we will need to include some singular fibers to projectively compactify a bundle of elliptic curves other than a torsor of a trivial fundle.

IV.2. Abstract theory.

Let $b \in \mathbb{P}^2$ be a point, and let $C \subset \mathbb{P}^2$ be a curve of degree four which does not pass through b. Consider the the pencil F_1 of projective lines through b considered disjoint from each other (a ruled surface), and its map to $\mathbb{P}^1$ viewed as the set of projective lines through b. Take as M the projective line with those points deleted which correspond to lines that fail to meet C transversely. Take as $\phi : \pi_1(M, m) \to S_4$ the monodromy action on the four points of intersection of m (viewed as a line) with C. Let $\widetilde{M}$ be the connected covering whose Galois group is the image $G = \phi(\pi_1(M, m)) \subset S_4/K_4$. The G-equivariant period map $\lambda : \widetilde{M} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ extends to a finite map from the completion of $\widetilde{M}$ to $\mathbb{P}^1$. Let L be the line bundle on F_1 whose section sheaf is the dual of the defining ideal of the curve C in F_1. Since this has class divisible by two, there is a line bundle N such that $N^{\otimes 2} = L$. Let s be a section of L whose zero variety defines C, which is the union of the four lines.

4. Corollary. The inverse image in N of the image $s(F_1)$ in L under the tensor square map is a complete projective variety which is a compactification of the bundle of elliptic curves defined by the period map λ.

IV.3. A double cover of $\mathbb{P}^2$.

Let's start slowly. We begin with the projective plane with given homogeneous coordinates $[x : y : z]$. We choose four general lines; up to automorphisms there is only one choice, and so we are free to choose the the lines defined by the equations

$$\begin{aligned} -2x + y - z &= 0 \\ x + y - z &= 0 \\ x - 2y - z &= 0 \\ -z &= 0 \end{aligned}.$$

Next, instead of considering the curve which is doubly branched over $\mathbb{P}^1$ at four *points*, we consider the surface which is doubly branched over $\mathbb{P}^2$ over these four *lines*. Now the inverse image of a general line is an elliptic curve in the surface.

Thus we consider the surface which is doubly branched over $\mathbb{P}^2$ along these lines. It has six nodes which can be resolved resulting in six pairs of Riemann spheres with each pair crossing at two points, all with normal degree -2.

The complement of a single point of $\mathbb{P}^2$ such as $[0 : 0 : 1]$ can be given the structure of a line bundle. Let M be a line bundle of degree one on $\mathbb{P}^1$. There is an open embedding

$$M \to \mathbb{P}^2$$

with image the complement of the point $[0 : 0 : 1]$, which can be defined like this: label two basic global sections of M with the names x, y, and for each pair of complex constants a, b map the global section $ax + by$ of M to the line in $\mathbb{P}^2$, which does not pass through $[0 : 0 : 1]$, which is defined by the equation $z = ax + by$. There is one point on this line for each value of the ratio $[x : y]$ and each section of M maps isomorphically to a line in $\mathbb{P}^2$ not passing through $[0 : 0 : 1]$.

In this way, we can view our four lines as really being sections of M.

We can construct the double cover of $\mathbb{P}^2$ simply like this. Call our four lines L_1, L_2, L_3, L_4 and consider a line bundle L on $\mathbb{P}^2$

with a section s such that the intersection of the image of s with the zero section equals the union of the four lines

$$s(\mathbb{P}^2) \cap \mathbb{P}^2 = L_1 \cup L_2 \cup L_3 \cup L_4$$

and is transverse except at the six points where two of the lines meet. The section sheaf of $\mathcal{L}$ of L is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(L_1 + L_2 + L_3 + L_4)$, the isomorphism given on local sections as

$$\mathcal{O}_{\mathbb{P}^2}(L_1 + L_2 + L_3 + L_4) \to \mathcal{L}$$

$$r \mapsto rs.$$

There is a line bundle N with $N^{\otimes 2} = L$ and the inverse image of $s(\mathbb{P}^2)$ under tensor square

$$N \to N^{\otimes 2} \cong L$$

is our surface with six nodes.

Let's name our four sections

$$\begin{aligned} e_1 &= -2x + y \\ e_2 &= x + y \\ e_3 &= x - 2y \\ e_4 &= 0 \end{aligned} .$$

Since e_4 and $e_1 + e_2 + e_3$ are zero, the tensor equation simplifies

$$v^{\otimes 2} = z^4 + (e_1 e_2 + e_1 e_3 + e_2 e_3) z^2 - e_1 e_2 e_3 z. \quad (1)$$

The extra factor of z on the right appears to be a slight correction of a mistake by Weierstrass; he had taken e_4 to be the exceptional section, and the surface could not be compactified because the divisor class of the sum of the four lines was not even.

Take our original vector bundle of degree 1 on $\mathbb{P}^1$, of which we've labelled two basic sections x and y, to be one which depends naturally on a choice of $0, 1, \infty$. Namely, we take the line bundle whose local sections are the local holomorphic one-forms with at worst simple (=logarithmic) poles of degree one at those three points. This line bundle extends the line bundle M_2 on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ whose sections are modular forms of weight 2 for $\Gamma(2)$. Hence we may take

$$\begin{aligned} x &= \frac{\pi^2}{3}\theta(0,\tau)^4 d\tau \\ y &= \frac{\pi^2}{3}\theta(0,1+\tau)^4 d\tau \end{aligned}.$$

The map which converts a modular form to a meromorphic one-form with at worst simple poles at $\{0, 1, \infty\}$, both locally and globally, is the one which is represented symbolically by appending $d\tau$. As I've explained more carefully elsewhere, the multiplication occurs as a product of a zero form with a one-form on $\mathbb{H}$ whereas on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, the multi-valued form $d\tau$ has a simple pole at all three points.

Let's use the letter M_2 to refer to this vector bundle of degree 1 on $\mathbb{P}^1$ whose restriction to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ has as its global sections the weakly modular forms of weight two for $\Gamma(2)$, and as its global sections the actual modular forms of weight two for $\Gamma(2)$.

The projection

$$\mathbb{P}^2 \setminus \{[0:0:1]\} \to \mathbb{P}^1$$
$$[x:y:z] \mapsto [x:y]$$

is a line bundle projection, and the section sheaf of the vector bundle has isomorphism type $\mathbb{O}_{\mathbb{P}^1}(1)$, same as the section sheaf of M_2. For each pair of complex numbers a, b, the line with equation

$$0 = z - ax - by$$

not passing through $[0:0:1]$ is a section of the line bundle, and we may quite simply think that the equation asserts that the fiber coordinate z must equal the section $ax + by$ of $\mathcal{O}_{\mathbb{P}^1}(1)$.

At the same time, the right side of this *equation* of the line is a global section of $\mathcal{O}_{\mathbb{P}^2}(1)$ which defines the same line in $\mathbb{P}^2$ by its intersection with the zero section.

Our equation (1) describes a section v of $\mathbb{O}_{\mathbb{P}^1}(2)$ whose tensor square equals the product of the four sections

$$(z - e_1)(z - e_2)(z - e_3)(z - e_4)$$

a section of $\mathcal{O}_{\mathbb{P}^2}(4)$ defining the union of the four lines.

In the section after next, we'll return to looking at the double cover of $\mathbb{P}^2$. We'll start to consider the lines in the pencil of lines through $[0 : 0 : 1]$ to be disjoint, thus resolving the indeterminacy of the map to $\mathbb{P}^1$. This inserts two rational curves with normal bundle degree -1 into the double cover of $\mathbb{P}^2$. The new rational curves just map to points of $\mathbb{P}^2$ but each maps isomorphically to $\mathbb{P}^1$. The inverse image in the double cover of each line in the pencil which doesn't pass through an intersection point of the four lines is an elliptic curve. Before resolving the indeterminacy, all such elliptic curves meet at just two points. After resolving the indeterminacy, the elliptic curves become disjoint and the two points are resolved to two exceptional lines.

First let's include a discussion of Weierstrass' function.

IV.4. Weierstrass' $\wp$ function

Weierstrass' relation for the $\wp$ function

$$(\frac{\partial}{\partial w}\wp(w,\tau))^2 = 4(\wp(w,\tau)^3+(e_1e_2+e_1e_3+e_2e_3)\wp(w,\tau)-e_1e_2e_3) \quad (2)$$

does still hold.

Although (1) and (2) appear to be equations of different degrees, we will need to relate them.

Recall $T : \mathbb{H} \to \mathbb{H}$ is our transformation $T(\tau) = \tau + 1$. Whenever $f : \mathbb{H} \to \mathbb{C}$ is a function, write f^T to be the function defined by $f^T(\tau) = f(T^{-1}\tau)$. We also define the 'coboundary' $i(f) = f - f^T$. Starting with

$$A(w,\tau) = \frac{\theta(w,\tau)^2}{\theta(0,\tau)^2}$$

and

$$g(\tau) = \theta(0, 1+\tau)^4 - 2\theta(0,\tau)^4$$

5. Definition. The $\wp$ function can be defined to be $\pi^2/3$ times the eigenfunction for the action of multiplying by g on i, that is,

$$\wp(w,\tau) = \frac{\pi^2}{3}\frac{i(gA)}{i(A)}$$

.

For fixed τ, in the elliptic curve $\mathbb{C}$ modulo translation by 1 and τ, the rational functions which express the linear equivalence between the divisor of order two at the point $\tau/2$ and the divisor of order two at the point $1 + \tau/2$ of order two curve specified by τ) is a constant a multiple of

$$\frac{\theta(w,\tau)^2}{\theta(0,\tau)^2}.$$

Choose the constant multiple (depending on τ) to be

$$\frac{\theta(w,\tau)^2\theta(0+\frac{1}{2},\tau)^2}{\theta(0,\tau)^2\theta(w+\frac{1}{2},\tau)^2}$$

Under the transformation $(w,\tau) \mapsto (w/(2\tau+1), \tau/(2\tau+1))$ the equation $w = \tau/2$ defining the pole is unaffected, and the equation $w = 1/2 + \tau/2$ defining the zero is affected by adding an integer multiple of τ to w. The resulting function is therefore invariant under $\Gamma(2)$ modulo scalar multiplications. It is in fact invariant as can be checked on the two generators of $\Gamma(2)$.

Since $\theta(w+1/2,\tau) = \theta(w,\tau+1)$ the ratio

$$R = \frac{A^T}{A} = \frac{\theta(w,1+\tau)^2\theta(0,\tau)^2}{\theta(0,1+\tau)^2\theta(w,\tau)^2}$$

is therefore an invariant meromorphic function, and we obtain the pair of of $\mathbb{Z}^2 \rtimes \Gamma(2)$-invariant meromorphic coefficients to express $\wp(z,\tau)$ as a linear combination of basic one-forms x and y

$$\wp(w,\tau)d\tau = \frac{1}{R-1}((R+2)x - (2R+1)y).$$

We can write this

$$\wp(w,\tau)i(A) = \frac{\pi^2}{3}i(gA).$$

From the definition of i this is

$$\wp(w,\tau)(A - A^T) = gA - (gA)^T.$$

This expands out to be

$$\wp(w,\tau)(\frac{\theta(w,\tau)^2}{\theta(0,\tau)^2} - \frac{\theta(w,1+\tau)^2}{\theta(0,1+\tau)^2})$$

$$= \frac{\pi^2}{3}((\theta(0,1+\tau)^4 - 2\theta(0,\tau)^4)\frac{\theta(w,\tau)^2}{\theta(0,\tau)^2} - (\theta(0,\tau)^4 - 2\theta(0,1+\tau)^4)\frac{\theta(w,1+\tau)^2}{\theta(0,1+\tau)^2}).$$

The one-form

$$\wp(w,\tau)d\tau$$

on $\mathbb{C} \times \mathbb{H}$ is a $\mathbb{Z}^2 \rtimes \Gamma(2)$-invariant one-form double pole on $0 \times \mathbb{H}$ and various zeroes elsewhere.

By definition 5, the divisor of this meromorphic one-form compares the "translation" invariant subvariety of A with that of the gA. The restriction of gA to the divisor of zeroes of our one-form is invariant under T and the restriction of A to the divisor of poles of our one-form is invariant under T. The divisor of this one-form is also, hence, invariant under T.

The deRham differential of $\wp(w,\tau)d\tau$ is a differential two-form; we can calculate it in the coordinates (w,τ) on $\mathbb{C} \times \mathbb{H}$ as

$$d\wp \wedge d\tau = \frac{\partial}{\partial w}\wp(w,\tau)dw \wedge d\tau$$

and it too is $\mathbb{Z}^2 \rtimes \Gamma(2)$ invariant.

Weierstrass' relation concerns two line bundles. One is the second tensor power of the second exterior power of the cotangent bundle, let us say of $\mathbb{C} \times \mathbb{H}$, and concerns the global section

$$(d\wp \wedge d\tau)^{\otimes 2}$$

of that line bundle. The other is the pullback of of the line bundle $M_3 = M_1^{\otimes 3}$ viewed as a $\Gamma(2)$ equivariant line bundle on $\mathbb{H}$ along the second projection of $\mathbb{C} \times \mathbb{H}$, and concerns the section

$$(\wp(w,\tau)d\tau - e_1)(\wp(w,\tau)d\tau - e_2)(\wp(w,\tau)d\tau - e_3).$$

Weierstrass' relation implies (and follows from) the condition that if we write the first section as a meromorphic function times the basic tensor $(dw \wedge d\tau)^{\otimes 2}$ and if we write the second as a meromorphic function times the basic tensor $d\tau^{\otimes 3}$, the two coefficient functions will be identical.

From the equivariant isomorphisms

$$\Lambda^2\Omega_{\mathbb{C}\times\mathbb{H}} \cong \Omega_{(\mathbb{C}\times\mathbb{H})/\mathbb{H}} \otimes p_2^*\Omega_{\mathbb{H}}$$

and

$$\Omega^{\otimes 2}_{(\mathbb{C}\times\mathbb{H})/\mathbb{H}} \cong p_2^*\Omega_{\mathbb{H}}$$

we can produce an equivariant isomorphism ϕ

$$\phi : \Lambda^2 \Omega^{\otimes 2}_{\mathbb{C}\times\mathbb{H}} \cong p_2^* \Omega^{\otimes 3}_{\mathbb{H}}.$$

Weierstrass' relation thus identifies a pair of global sections which correspond with one another under the equivariant isomorphism ϕ between the second tensor power of one line bundle and the third tensor power of the other

$$\phi((d\wp \wedge d\tau)^{\otimes 2}) = (\wp d\tau - e_1) \otimes (\wp d\tau - e_2) \otimes (\wp d\tau - e_3).$$

If we set

$$v = \sqrt{\wp} dw \otimes (d\wp \wedge d\tau)$$

we obtain an equation which does match the restriction of (1) to an open subset of $\mathbb{P}^2$

$$(\psi \otimes \phi)(v^{\otimes 2}) = (\wp d\tau - e_1) \otimes (\wp d\tau - e_2) \otimes (\wp d\tau - e_3) \otimes (\wp d\tau - e_4)$$

with $e_4 = 0$, where ψ is our equivariant isomorphism $\Omega^{\otimes 2}_{(\mathbb{C}\times\mathbb{H})/\mathbb{H}} \to p_2^* \Omega_{\mathbb{H}}$.

IV.5. The elliptic surface

We may take our rational structural map to be $\lambda = \frac{(e_1-e_3)}{(e_1-e_2)}$. As a map whose domain is our double cover of $\mathbb{P}^2$ λ is only a rational map; once we resolve the indeterminacy of λ then we have our elliptic surface over $\mathbb{P}^1$ which still has six nodes.

The cross-ratio

$$\gamma = \frac{(e_3-e_2)(e_1-e_4)}{(e_1-e_2)(e_3-e_4)}$$

factorizes through the structural map defining a quadratic period map $\mathbb{P}^1 \to \mathbb{P}^1$.

Since we've taken the e_i to be functions of τ then λ is also a function τ, it is precisely the classical lambda function.

The period map factors through the structural map, as we mentioned, and equals the quadratic map $\mathbb{P}^1 \to \mathbb{P}^1$ which is given

$$\gamma(\lambda) = \frac{1-\lambda^2}{1-2\lambda}.$$

The degree-two map $\lambda \mapsto \gamma(\lambda)$ sends the values of λ which parametrize elements of the pencil which meet a crossing point of two of the same lines, which are

$$-1, 0, \frac{1}{2}, 1, 2, \infty,$$

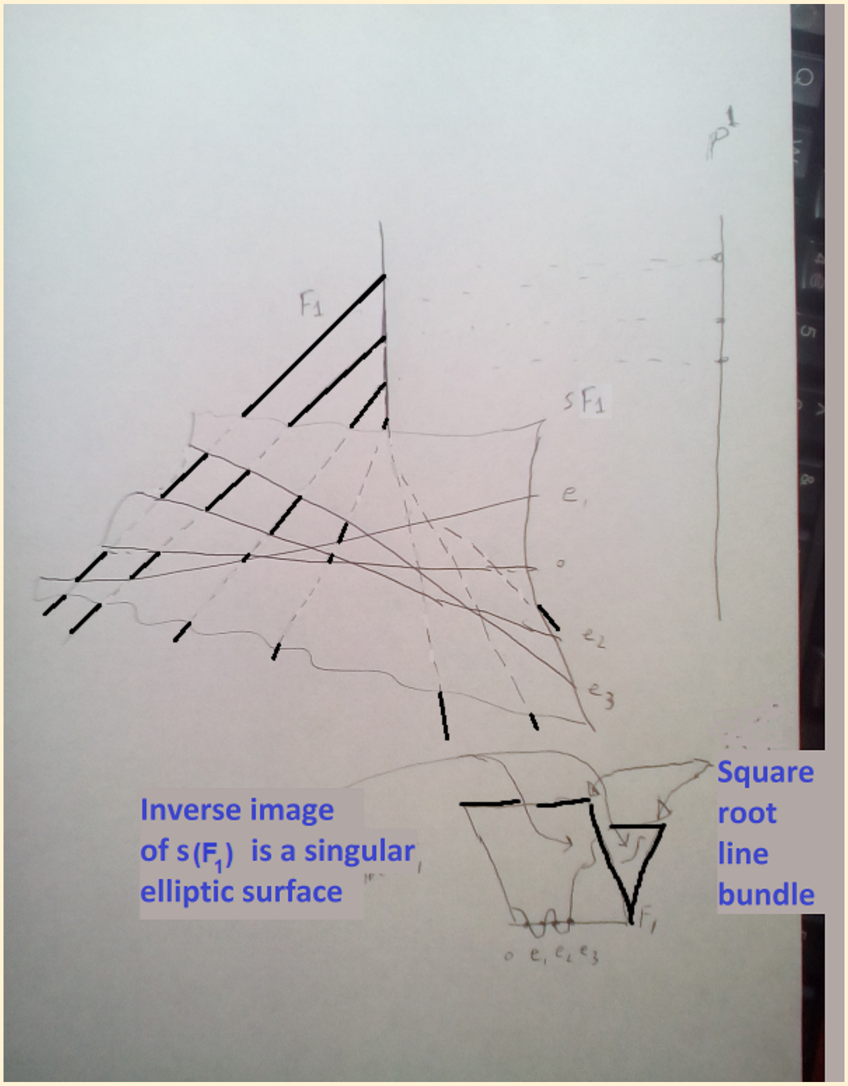
in order, to

$$0, 1, \infty, 0, 1, \infty.$$

The two branching points of the quadratic period map are when $\lambda(\tau) = e^{\pm 2i\pi/6}$, where $\tau = e^{i\pi/2 \mp i\pi/6}$.

The six singular fibers of the resulting compact projective surface have a node each, which is a node in the ambient surface. Once the six nodes are resolved, each singular fiber is a union of two rational curves intersecting at two points each, and each with normal bundle of degree -2.

The unresolved elliptic surface S is the inverse image under tensor square $N \to N^{\otimes 2} \cong L$ of the section image $s(F_1) \subset L$ with L being the line bundle whose section sheaf is dual to the defining ideal $\mathcal{I}$ of the union of four lines in F_1.



The nontrivial Galois automorphism of the map $\lambda \mapsto \frac{1-\lambda^2}{1-2\lambda}$ is easily calculated, if we say $\lambda \mapsto c$ then λ satisfies that $1 - \lambda^2 = c(1 - 2\lambda)$; the two solutions of this equation add to $2c$ so the other solution is $\frac{\lambda-2}{2\lambda-1}$.

Rather than preserving the image of the classical lambda function, this automorphism interchanges 0 and 2 and interchanges 1 and -1, and interchanges ∞ and $1/2$. The period map describes a Galois cover of $\mathbb{P}^1$ branched at two points, but it is not possible to lift the Galois automorphism to any automorphism of $\mathbb{H}$ because it sends three interior points to ideal points.

The cover $\widetilde{M} \to M$ is just trivial (six disjoint copies of M itself for $M = \mathbb{P}^1 \setminus \{0, 1, \infty\}$), and a connected component is just a copy of M itself, so there is no need to consider equivariance.

The period map itself is a degree-two Galois cover; the nontrivial Galois automorphism of $\mathbb{P}^1$ induces a *non-algebraic* automorphism of our surface S once we view S as the pullback of a non-algebraic surface along the period map. Our surface S is *analytically* a branched cover of degree two, branched on two smooth fibers. While it is *algebraically* a branched cover of the scroll branched along four Riemann spheres.

The connected components of the Picard group of our surface are a free abelian group of rank 10. It is rationally the same as $H^1(S, \Omega_S)$ and since the higher derived functors of pushing forwards along the branched cover to the scroll are trivial, there is a basis conssting of the two classes which span Pic of the scroll, and a rational basis of 8 anti-invariant classes. Six of these merely had to do with resolving the six singular points. Two remaining anti-invariant classes remain to be understood.

IV.6. Extension of the period map.

Each of the lines L_1, L_2, L_3, L_4, since it is a section of a line bundle over $\mathbb{P}^1$, has a determined isomorphism with $\mathbb{P}^1$. Then the period map $\gamma : \mathbb{P}^1 \to \mathbb{P}^1$ induces a map on each line separately.

Let's choose the line L_4 in particular. We identify L_4 with $\mathbb{P}^1$, and in our coordinates $[x : y : z]$ for $\mathbb{P}^2$, if we think of $\mathbb{P}^2$ as the $[x : y : 1]$ plane union the line at infinity with equation $z = 0$, we are identifying $\mathbb{P}^1$ with the line at infinity in the usual way.

Next, if we interpret $\mathbb{P}^1$ as the cone on L_4 with smooth cone point $[0 : 0 : 1]$, the period map induces the the Galois automorphism

$$\mathbb{P}^2 \cong \mathbb{P}^2$$

$$[x : y : z] \mapsto [x - 2y : 2x - y : -\sqrt{-3}z].$$

The induced map on the $[x : y : 1]$ plane is given by the matrix

$$\begin{pmatrix} \frac{-1}{\sqrt{-3}} & \frac{2}{\sqrt{-3}} & 0 \\ \frac{-2}{\sqrt{-3}} & \frac{1}{\sqrt{-3}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The closure of the $+1$ eigenspace is a fixed Riemann sphere in $\mathbb{P}^2$ and the closure of the -1 eigenspace is a Riemann sphere which is preserved but not fixed, only the point $[0 : 0 : 1]$ on that line and its intersection with the line L_4 at infinity are the two fixed points belonging to that Riemann sphere, and the Galois automorphism acts on it by an involution.

The quotient of $\mathbb{P}^2$ by the action of the Galois automorphism is a weighted projective space of the isomorphism type of $\mathbb{P}(1, 2, 1)$. It has a node which is the image of the point where the two lines which are preserved but not fixed pointwise intersect.

IV.7. Analytic parametrization.

Consider the map

$$(\mathbb{Z} \rtimes \Gamma(2))\backslash(\mathbb{C} \times \mathbb{H}) \to \mathbb{P}^2$$

$$(w, \tau) \mapsto [\pi^2\theta(0,\tau)^4 : \pi^2\theta(0,1+\tau)^4 : 3\wp(w,\tau)].$$

coming from our discussion of the $\wp$ function.

Since the order of the pole of $\wp$ is two, or more clearly since $\wp(w,\tau) = \wp(-w,\tau)$, once we compose with the non-Galois covering map $M_{-1} \to (\mathbb{Z} \rtimes \Gamma(2))\backslash(\mathbb{C} \times \mathbb{H})$ the map factorizes through tensor square $M_{-1} \to M_{-2}$ to the total space of the line bundle M_{-2} on $\mathbb{P}^1$. In other words, it would have been better to use w^2 as a variable, or use the action of weight -2 instead of the action of weight -1 in defining the quotient $(\mathbb{Z} \rtimes \Gamma(2))\backslash(\mathbb{C} \times \mathbb{H})$.

With that change made, we now look at the zero section. The zero section $0 \times \mathbb{H}$ maps to the single point $[0 : 0 : 1]$ because of the pole of the $\wp$ function, therefore our map lifts to a map to the scroll F_1. The lifted map is a two-fold branched cover now, with image the open subset of F_1 which is the complement of the line L_4 union the lines $V(x), V(y), V(x-y)$, and branched over $(L_1 \cup L_2 \cup L_3) \setminus (L_4 \cup V(x-y) \cup V(x) \cup V(y))$.

Thus, our original map to $\mathbb{P}^2$ factorizes through an analytic parametrization of an open subset of our elliptic surface with six nodes.

The lines in $\mathbb{C} \times \mathbb{H}$ where the second coordinate τ is in the orbit of $\{i, 1+i, \frac{1+i}{2}\}$ map to the three lines through $[0 : 0 : 1]$ which are asymptotic to the three branching lines which meet our open subset, namely $L_1 = V(-2x+y-z), L_2 = V(x+y-z), L_3 = V(x-2y-z)$. That is to say, the lines through $[0 : 0 : 1]$ meet our 'line at infinity' L_4 at its crossing points with the three lines L_1, L_2, L_3 which do *not* pass through $[0 : 0 : 1]$

6. Remark. It might be more symmetrical if we delete the orbits of $i, 1+i, \frac{1+i}{2}$ and then our parametrization is of a branched double covering of the scroll with the fibers over $-1, 0, \frac{1}{2}, 1, 2, \infty$ and the section L_4 all deleted.

The open subset of $\mathbb{P}^2$ which is the base of the double covering is now preserved under the automorphism of $\mathbb{P}^2$ which extends Galois automorphism of $\mathbb{P}^1 = L_4$ of the period map and fixes $[0:0:1]$. And this copy of $\mathbb{P}^2$ in turn is a double branched cover, branched at the two lines from $[0:0:1]$ to two points of L_4 with coordinates sixth roots of unity.

Elsewhere we've mentioned that the real points of $\mathbb{P}^1$ are covered by the ideal triangle in $\mathbb{H}$ consisting of its intersection with the arcs $Re(\tau) = 0, Re(\tau) = 1$, and $|\tau - 1/2| = 1/2$. The deleted fibers consist of the endpoints of those arcs, and the points $i, \frac{1+i}{2}, 1+i$ which serve as midpoints of the same three arcs.

As a map on the real points of $\mathbb{P}^1$, the period map is a regular double cover, sending the six points in order which are endoints or midpoints, all to the three endpoints.

A simple way of understanding the relation between τ and the period map is to notice that by the definition of e_1, e_2, e_3, the ratio $\frac{e_1-e_3}{e_1-e-2} = 1 - \frac{y}{x}$ is the classical lambda function (where all of e_1, e_2, e_2, x, y are functions of τ. This is a cross-ratio of e_1, e_2, e_3, ∞.

The elliptic curve which is the restriction of our branched cover of $\mathbb{P}^2$ to the line through $[0:0:1]$ with a particular slope $[x : y]$ corresponds to the branch points where z equals e_1, e_2, e_3, e_4 with $e_4 = 0$.

Crucially, at least the way we have chosen this parametrization, the variable τ to which we apply $\wp$ is not a period ratio of the elliptic curve which is the image of the line $\mathbb{C} \times \{\tau\}$.

If we apply the classical lambda function to a period ratio of this elliptic curve, the result will be $\gamma(\lambda(\tau))$.

IV.8 Parametrization of the extended period map

Let's choose as a fundamental domain for the action of $\Gamma(2)$, which is the same as for the action of $\langle T^2, ST^2S, \rangle$ since minus the identity acts trivially, the pair of ideal triangles in $\mathbb{H}$ with ideal vertex set $\{0, 1, i\infty\}$ and $\{1, 2, i\infty\}$. Let's call this fundamental domain D. The composite

$$Interior(D) \subset \mathbb{H} \to \Gamma(2) \setminus \mathbb{H} \to \mathbb{P}^1 \setminus \{0, 1, \infty\} \subset \mathbb{P}^1$$

extends to a map

$$D \to \mathbb{P}^1$$

by which we can interpret $\mathbb{P}^1$ as an identification space, made by gluings on the boundary of D.

We can lift the period map $\gamma : \mathbb{P}^1 \to \mathbb{P}^1$ to a map $\eta : D \to D$ such that the diagram commutes

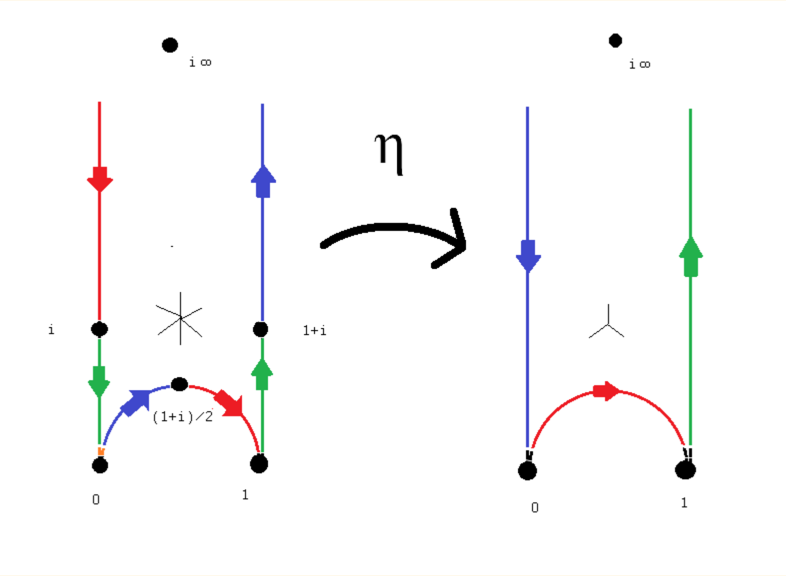
$$\begin{array}{lcl} D & \to & \mathbb{P}^1 \\ \downarrow \eta & & \downarrow\, \gamma \\ D & \to & \mathbb{P}^1 \end{array}.$$

On the ideal triangle in ideal vertices $0, 1, i\infty$, consider in order in the boundary of this triangle the six points $0, \frac{1+i}{2}, 1, 1+i, i\infty, i$. Consider the holomorphic map which sends the six geodesic segments in order between these points to the three edges of the same triangle, so our map on vertices is

$$\begin{array}{ccc} i\infty & \mapsto & 0 \\ i & \mapsto & 1 \\ 0 & \mapsto & \infty \\ \frac{1+i}{2} & \mapsto & 0 \\ 1 & \mapsto & 1 \\ 1+i & \mapsto & \infty \end{array}.$$

This describes a branched conformal map, let us call it η. It is a branched double cover of the ideal triangle, with the branch point of order two when $\tau = e^{2i\pi/6}$. It has the property that for our period map γ

$$\gamma(\lambda(\tau)) = \lambda(\eta(\tau)).$$



The map and the formula extend by symmetry to the second ideal triangle needed to cover $\mathbb{P}^1$ and the map has a second branch point in the second ideal triangle.

Now there is a unique way to complete this commutative diagram

$$\begin{array}{ccc} \mathbb{C} \times D & \rightarrow & \mathbb{P}^2 \\ \downarrow 1 \times \eta & & \downarrow \\ \mathbb{C} \times D & \dashrightarrow & V \end{array}$$

where the top vertical map is

$$\mathbb{C} \times D \to \mathbb{P}^2$$

$$(w, \tau) \mapsto [\pi^2 \theta(0, \tau)^4 : \pi^2 \theta(0, 1 + \tau)^4 : 3\wp(w, \tau)],$$

where $V = \langle \sigma \rangle \backslash \mathbb{P}^2$, where σ is our order-two Galois automorphism, and the right vertical map is reduction modulo σ. With the correct boundary gluings, the copy of $\mathbb{C} \times D$ on the upper left becomes homeomorphic to the total space of the line bundle M_{-2} on $\mathbb{P}^1$.

I believe that the map factorizes through our elliptic surface S, and M_{-2} is a non-Galois cover of the regular locus of S (the complement of the set of six nodes).

The zero section and one other rational section of M_{-2} must then map to the intersection with the regular locus of S of the two exceptional curves in S whose disjoint union is the inverse image of $[0:0:1] \in \mathbb{P}^2$.

The poles of the rational section may occur only at the six special points of $\mathbb{P}^1$ which are $-1, 0, 1/2, 1, 2, \infty$ such that the corresponding lattice in the fiber over that point degenerates, and the fiber modulo the lattice is not a complete variety – matching how six of the fibers of $S \to \mathbb{P}^1$ become non-complete when the nodes are deleted from S.

The Galois symmetry of S over $\mathbb{P}^2$ in interchanging the two exceptional curves in S corresponds to interchanging the zero section with this other rational section and does not respect the structure of M_{-2} as a line bundle.

Thinking now about the other Galois automorphism, the non-trivial Galois automorphism of $\mathbb{P}^2$ over V, each projective line through $[0:0:1]$ in $\mathbb{P}^2$ which does not pass through a crossing point among L_1, L_2, L_3, L_4 has four points where it meets the L_i, and four points where it meets the transforms of L_1, L_2, L_3, L_4 via that automorphism. The automorphism preserves L_4 the way we have chosen it, and so when we look at the image of such a line in V, it has as special points the point where it meets the image of L_4, but now two points where it meets the image of each of L_1, L_2, L_3.

By construction, there must be an automorphism bringing one set of three to the other while fixing the fourth.

When we write $c = 1 - \lambda = \frac{y}{x}$ then when we fix c the line in the affine plane defined by the equation $y = cx$ meets L_1, L_2, L_3, L_4 when x satisfies

$$1 + 2x - y = 0$$

$$1 - x - y = 0$$

$$1 - x + 2y = 0$$

so $x = \frac{1}{c-2}, \frac{1}{c+1}, \frac{1}{1-2c}$. The points of intersection determine these three values of x together with $x = \infty$.

To apply our automorphism replace x and y by $\frac{-1}{\sqrt{-3}(x-2y)}$ and $\frac{-1}{\sqrt{-3}}(2x - y)$ so the equations above become

$$1 + \frac{-2}{\sqrt{-3}}(x - 2y) + \frac{1}{\sqrt{-3}}(2x - y) = 0$$

$$1 + \frac{1}{\sqrt{-3}}(x - 2y) + \frac{1}{\sqrt{-3}}(2x - y) = 0$$

$$1 + \frac{1}{\sqrt{-3}}(x - 2y) + \frac{-2}{\sqrt{-3}}(2x - y) = 0.$$

Now when $y = cx$ these simplify to

$$x = \frac{1}{\sqrt{-3}c}$$

$$x = \frac{1}{\sqrt{-3}(1 - c)}$$

$$x = \frac{-1}{\sqrt{-3}}$$

The points of intersection now determine these values of x and ∞. If we reciprocate all eight values, and ignore the coefficient of $\sqrt{-3}$ as we may, we obtain

$$c - 2, c + 1, 1 - 2c, 0$$

and

$$c, 1 - c, -1, 0.$$

The cross-ratios are reciprocals

$$\frac{((-1) - (c))(1 - c)}{((1 - c) - (c))(-1)} = \frac{1 - c^2}{1 - 2c}$$

$$\frac{((c + 1) - (c - 2))(1 - 2c)}{((1 - 2c) - (c - 2))(c + 1)} = \frac{1 - 2c}{1 - c^2}.$$

so that when we for example interchange the middle two entries in either list they become the same.

What this means about the variety V is that the image in V of each general line in $\mathbb{P}^2$ through $[0 : 0 : 1]$ has two different distinguished subsets of three points, and the deleted Riemann sphere is isomorphic if the point at infinity together with either set of three points is separately deleted; yet if we used the ordering of the lines L_1, L_2, L_3 to order the points upon the lines, the isomorphism would not preserve this ordering.

It is also true, that for the projective line in V, we may take a doubly branched cover over the 'point at infinity' together with either set of three points, and the result will be an elliptic curve, isomorphic regardless of which case we choose. But the isomorphism is not the identity, and if it is chosen to preserve the set four points (a torsor for the Klein four-group in the Jacobian), it induces an odd permutation of those points.

V.9. Covers, connections and compactifications

We have seem that for the action of degree -2, the quotient $\Gamma(2)\backslash(\mathbb{C}\times\mathbb{H})$ has the structure of a line bundle on $\mathbb{P}^1\setminus\{0,1,\infty\}$ which makes it isomorphic to the total space of M_{-2}. This is a non-Galois cover of the quotient by the larger group $(\mathbb{Z}^2\rtimes\Gamma(2))\backslash(\mathbb{C}\times\mathbb{H})$ and this quotient space embeds as an open subset of our elliptic surface S with six nodes, such that the image double covers the complement in the scroll F_1 of the line L_4 union the six projective lines through $[0:0:1]$ which meet L_4 at $-1,0,\frac{1}{2},1,2,\infty$.

The order-two automorphism

$$\mathbb{P}^2\to\mathbb{P}^2$$
$$[x:y:z]\mapsto[x-2y:2x-y:-\sqrt{-3}z]$$

preserves the union of this set of lines, and fixes $[0:0:1]$ so that in the lift to an automorphism of the scroll F_1 it fixes the exceptional line too.

The homogeneous coordinates x,y,z are, more rigorously, sections of M_2, that is, functions $M_{-2}\to\mathbb{C}$ which happen to be linear on the line fibers. The particular coordinate transformation we've written down, such as the 3×3 matrix which we wrote earlier, does describe an order-two automorphism of M_{-2} which is compatible with the line bundle projection and the order-two automorphism of the base $\mathbb{P}^1$ of the line bundle.

Our current question is this: Can we find a pair of commuting order-two automorphisms $(\mathbb{Z}^2\rtimes\Gamma(2))\backslash(\mathbb{C}\times\mathbb{H})$ where the action is the one of degree -2, such that the quotient modulo one automorphism is a copy of $(\mathbb{Z}^2\rtimes\Gamma(2))\backslash(\mathbb{C}\times\mathbb{H})$ with the action taken to be of degree -1, and the quotient modulo the other is the scroll F_1 with the six lines through $[0:0:1]$ which meet a crossing among L_1,L_2,L_3,L_4, as well as L_4 itself deleted, and such that when we reduce modulo both automorphisms we arrive at the complement in V of the images of those same projective lines, now being three projective lines through the image of $[0:0:1]$ and the projective line with a double branched cover by L_4.

V. Differential calculus on elliptic curves.

V.1. Fiberwise vector fields

Let $S \to M$ be a smooth holomorphic bundle of elliptic curves. Let L be the corresponding line bundle of Lie algebras on M. Let $\mathcal{S}$ be the coherent sheaf on M which consists of fiberwise vector fields on S which commute with addition by local sections of the corresponding Jacobian bundle $J \to M$.

7. Theorem. The sheaf $\mathcal{S}$ is naturally isomorphic with the sheaf of sections of L.

We want to consider more general surfaces mapping to $\mathbb{P}^1$ with some singular fibers.

It is useful now to speak of the line bundle M_k on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ whose sections correspond to modular forms of weight k for our group $\langle T^2,\ ST^2S\rangle$ for k a positive or negative integer.

A technicality is that there are no global modular forms of weight 1 for the subgroup of $Sl_2(\mathbb{Z})$ which is the inverse image of $\Gamma(2) \subset PSl_1(\mathbb{Z})$. Instead, we lift $\Gamma(2)$ isomorphically to the subgroup generated by T^2, ST^2S where $T(\tau) = \tau + 2$, $S(\tau) = -1/\tau$ and we define M_k for all integers k to be the vector bundle on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ whose local sections are what are called 'weakly modular forms' of weight k for $\langle T^2, ST^2S\rangle$, that is, which locally satisfy the transformation law of weight k for that group. One way to see define what it means to be locally weakly modular is to directly construct the line bundle M_k for all k as $\mathbb{C} \times \mathbb{H}$ modulo the action of $\langle T^2, ST^2S\rangle$ where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (w, \tau) = ((c\tau + d)^k w, \frac{a\tau + b}{c\tau + d}),$$

and define local weakly modular forms of weight k to be local sections of M_k.

The fiberwise vector fields on M_k have divisor the pullback to M_k of the divisor defining that very line bundle, namely k times a point up to linear equivalence, where k may be positive or negative, and local sections of the sheaf of fiberwise vector fields once restricted to the complement of the singular fibers, pull back on $\mathbb{C}\times\mathbb{H}$ to the $\Gamma(2)$-invariant vector fields of the form $g(\tau)\frac{\partial}{\partial w}$ where g is locally weakly modular of weight $-k$.

With finer analysis one might be able to show more in more general situations that such vector fields fix only the singular subscheme of each fiber, and it is this which forces the restriction on the singular fibers in Kodaira's classification.

Let's try to construct a fiberwise meromorphic vector field on S with a simple pole on the fiber over one of the branching points of the period map.

Consider the tensor square map $M_{-1} \to M_{-2}$, and we consider the extension of M_{-2} to a holomorphic line bundle on $\mathbb{P}^1$ of degree -1 therefore isomorphic to $\mathbb{O}(-1)$. The particular section $\frac{1}{4\pi^2\theta(0,\tau)^4}$ on $\mathbb{P}^1 \setminus \{0,1,\infty\}$ extends to a rational section of $\mathbb{O}(-1)$. The inverse image of the section image under tensor square is the 2-section of M_{-1} which we might write as $\frac{\pm 1}{2\pi\theta(0,\tau)^2}$.

The vector field $\frac{1}{2\pi\theta(0,\tau)^2}\frac{\partial}{\partial w}$ is invariant under $\mathbb{Z}^2\rtimes\Gamma(2)$ for the action of weight -1. Starting at the point at infinity in each elliptic curve, the paralellogram of 'time' $\{q2\pi\theta(0,\tau)^2 + r\tau 2\pi\theta(0,\tau)^2 : 0 \leq q, r < 1\}$ is a fundamental domain, covering each elliptic curve precisely once when τ is fixed, and when τ ranges over a fundamental domain for $\Gamma(2)$ this set covers the complement of the singular fibers in the elliptic surface with three singular fibers. For each fixed value of τ, the arc of 'time' from 0 to 1 double covers the arc in the real projective plane where $\lambda \in (-\infty, 0)$ and the arc of 'times' from 0 to τ double covers the arc in the real projective where $\lambda \in (1,\infty)$. The two double covering arcs meet transversely at the point at infinity in each elliptic curve fiber.

For our surface with 6 singular fibers, the pullback along the period map, we have the basic period formulas

$$2\int_{-\infty}^{0} \frac{dz}{\sqrt{z(z-1)(z-\gamma(\lambda))}} = 2\pi\tau\theta(0,\tau)^2$$

$$2\int_{1}^{\infty} \frac{dz}{\sqrt{z(z-1)(z-\gamma(\lambda))}} = 2\pi\theta(0,\tau)^2.$$

Twice the integral between two branch points expresses the integral around each basic closed loop in each elliptic curve fiber.

That is to say, the vector field on each elliptic curve fiber such that the directional derivative of z is $\sqrt{z(z-1)(z-\gamma(\lambda))}$ integrates to zero when time ranges from 0 to $2\pi\theta(0,\tau)^2$ or from 0 to $2\pi\tau\theta(0,\tau)^2$.

Let's choose the point where $\lambda = e^{2\pi i/6}$. The branching points of γ are fixed points of γ, so also $\gamma(\lambda) = e^{2\pi i/6}$.

We choose a modular form of weight one whose square has a simple zero at the point $\gamma(\lambda)$. The *classical* lambda function takes the value $e^{2\pi i/6}$ when $\tau = e^{2i\pi/6}$, and so we take as our (multivalued?) modular form, using the rule $1 + e^{2i\pi/3} = e^{i\pi/3}$ as

$$f(\tau) = \sqrt{\theta(0, e^{2i\pi/3})^4\theta(0, 1+\tau)^4 - \theta(0, e^{i\pi/3})^4\theta(0,\tau)^4}.$$

There is a constant $c \cong 1.03...$ which seems to be a positive real number, such that

$$\theta(0, e^{2\pi i/3})^4 = ce^{-i\pi/6}$$

$$\theta(0, e^{i\pi/3})^4 = ce^{i\pi/6}.$$

Thus

$$f(\tau) = c^{1/2}\sqrt{e^{11i\pi/6}(\theta(0, 1+\tau)^4 + e^{7i\pi/6}\theta(0,\tau)^4}.$$

The vector field

$$\frac{1}{f(\tau)}\frac{\partial}{\partial w}$$

on $\mathbb{C}\times\mathbb{H}$ is invariant under the action of $\Gamma(2)$ by which $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(w,\tau) = \frac{w}{c\tau+d}, \frac{a\tau+b}{c\tau+d})$, and defines a vector field on the bundle M_{-1} on $\mathbb{P}^1 \setminus \{0,1,\infty\}$ which has a simple pole on the fiber over the branching point of γ.

Since we denote the period map $\mathbb{P}^1 \to \mathbb{P}^1$ by γ, then the complement of the singular fibers in our surface S is the total space of $\gamma^* M_{-1}$, and our fiberwise vector field lifts to a vector field having a simple pole on one fiber.

Two basic periods for the elliptic curve over a point $\lambda \in \mathbb{C} \setminus \{0,1,\infty\} \subset \mathbb{P}^1 \setminus \{0,1,\infty\}$ are then obtained as follows: starting with λ we evaluate $\gamma(\lambda) = \frac{1-\lambda^2}{1-2\lambda}$. Then we define τ to be

$$\tau = \frac{\int_{-\infty}^{0} \frac{dz}{\sqrt{z(z-1)(z-\gamma(\lambda))}}}{\int_{1}^{\infty} \frac{dz}{\sqrt{z(z-1)(z-\gamma(\lambda))}}}.$$

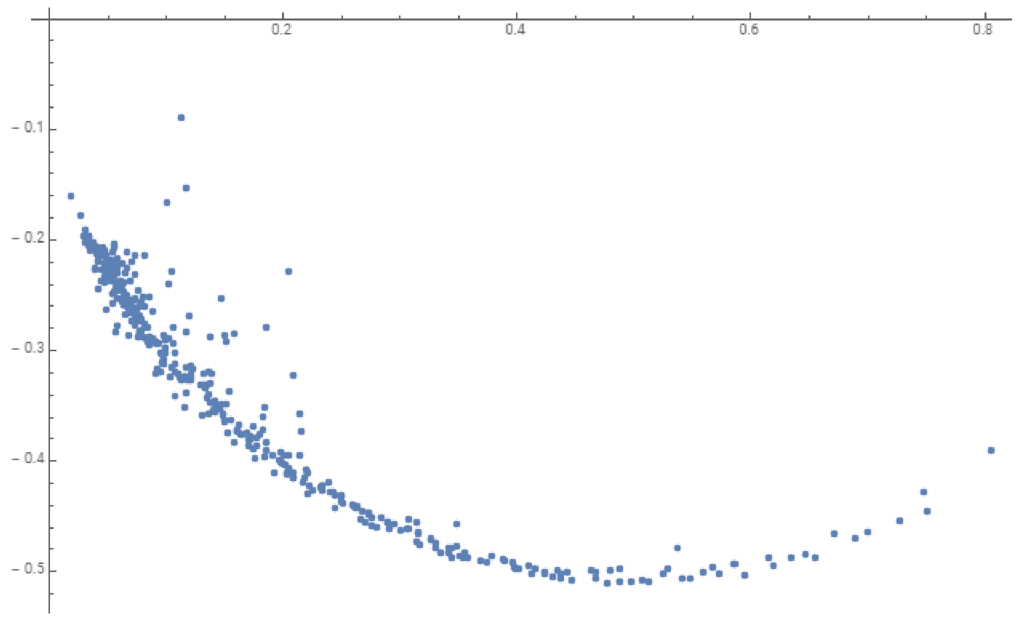
Then our two basic periods are

$$f(\tau), \tau f(\tau).$$

In an earlier section, we showed that as the variable τ goes around the ideal triangle in $\mathbb{H}$ with vertices $i\infty, 0, 1$ the corresponding period ratio $\eta(\tau)$ goes twice around the same ideal triangle. To verify the formulas above, let's use them to approximate the period ratio $\eta(\tau)$ when τ is various points in the interval from i to $i\infty$. After negating these and operating by $\Gamma(2)$ they belong to the circle of radius $1/2$ centered at $1/2$.

```
In[1494]:= g[l_] := (1 - l^2)/(1 - 2*l)
           PP[z_] := {Re[z], Im[z]}
           Int1[l_] := 2*NIntegrate[(z(z-1)(z-g[l]))^(-1/2), {z, -Infinity, 0}]
           Int2[l_] := 2*NIntegrate[(z(z-1)(z-g[l]))^(-1/2), {z, 1, Infinity}]
           ListPlot[Table[PP[Int1[ModularLambda[t*I/100]]/Int2[ModularLambda[t*I/100]]], {t, 101, 501}]]
```

Out[1498]=



It remains to extend our vector field to the compactification, something that will have to wait for a later section.

When we compare vector fields constructed before, versus after, reducing modulo the Galois automorphism of the period map, we must make the correct identification between the surfaces which are the total spaces of $M_{-1}^{\otimes 2}$ and M_{-2}.

V.2. Vector fields modulo fiberwise vector fields

We can analyze the vector fields on M_k for any (positive or negative) integer k. Letting h be the structural map $h : M_k \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$, we have

$$0 \to T_{M_k/(\mathbb{P}^1\setminus\{0,1,\infty\}} \to T_{M_k} \to h^* T_{\mathbb{P}^1\setminus\{0,1,\infty\}} \to 0$$

with

$$T_{M_k/(\mathbb{P}^1\setminus\{0,1,\infty\}} \cong h^* M_k$$

the pullback of M_k along its own structural map, and

$$h^* T_{\mathbb{P}^1\setminus\{0,1,\infty\}} \cong h^* M_2.$$

As a reminder about indexing, if we think of $\frac{\partial}{\partial\tau}$ as a multi-valued vector field which takes three simple zeroes on $\mathbb{P}^1$ and transforms as if it were a modular form of weight two, then we obtain a $\langle T^2, ST^2S\rangle$ invariant vector field by multiplying by a modular form of weight -2, viewed as a global *meromorphic* object, which will have a simple pole which is capable of cancelling one of the three zeroes up to linear equivalence. A global section of the sheaf of pulled-back vector fields modulo the fiberwise subsheaf, then, will be expressed in the form $f(\tau)\frac{\partial}{\partial\tau}$ where f is locally weakly modular of weight -2, and the whole vector field will have a zero locus equivalent to two line fibers of the line bundle M_k.

Although the elliptic surface with three singular fibers is not algebraic, we can construct the global sections of vector fields modulo fiberwise vector fields, they form a two-dimensional vector space with basis $\frac{1}{\theta(0,\tau)^4}\frac{\partial}{\partial\tau}$ and $\frac{1}{\theta(0,1+\tau)^4}\frac{\partial}{\partial\tau}$, and the same expressions are a basis of the two-dimensional vector space of vector fields modulo fiberwise vector fields for our algebraic surface S, as long as we understand that the variable τ here refers to the actual period ratio of the fibers, something which we'll have called $\eta(\tau)$.

We are interested in the relation between our algebraic surface S and a more naive construction of the elliptic fiber bundle over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ which has the non-Galois cover by M_{-1}

corresponding to the non-normal subgroup $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes \Gamma(2)$. The complement of the six singular fibers in S union the inverse image of the line L_4 is the branched cover of M_{-1} which is induced by the quadratic period map; that is, that it is the pullback of the structural map of the line bundle

$$M_{-1} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

with the period map

$$\mathbb{C} \setminus \{-1, 0, \frac{1}{2}, 1, 2, \infty\} \to \mathbb{C} \setminus \{0, 1\}$$

$$\lambda \mapsto \frac{1-\lambda^2}{1-2\lambda}.$$

Any vagueness in these definitions stems from earlier lack of rigor in speaking of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ which really doesn't make sense as notation, and should have been written $\mathbb{P}^1 \setminus \{[0:1], [1:1], [1:0]\}$.

Because the modular form $\theta(0, \tau)$ isn't zero at any interior points of the upper half plane, there is a global section of the sheaf of fiberwise vector fields, and a vector field which descends to a global vector field modulo fiberwise vector fields. These are, respectively, $\theta(0, \tau)^{-2} \frac{\partial}{\partial w}$ and $\theta(0, \tau)^{-4} \frac{\partial}{\partial \tau}$.

On our elliptic surface, the restriction of the sheaf of fiberwise holomorphic vector fields is the sheaf of holomorphic functions times $\theta(0, \tau)^{-2} \frac{\partial}{\partial w}$

And we know abstractly that any global meromorphic section of this sheaf has a divisor of poles equivalent to one fiber of $S \to \mathbb{P}^1$.

For the restriction of the sheaf of holomophic vector fields modulo fiberwise ones, we must consider that $\theta(0,\tau)^{-4}\frac{\partial}{\partial\tau}$ does not pull back through the branched cover. It is entirely the same issue if we forget about the w coordinate and consider the vector field as a vector field on $\mathbb{C}\setminus\{0,1\}$. Let's use the coordinatization where the ideal values $\tau = 0, i\infty, 1$ corresponds to $1, 0, \infty$. The coefficient $\theta(0,\tau)^{-4}$ cancels the zero at 1 and so our vector field extends to a vector field on $\mathbb{P}^1$ with a simple zero at 0 and ∞.

We must multiply by a rational function with a pole at 0 and ∞ and a simple zero at both branching points before we can lift the vector field. Then we obtain a global section on S of vector fields modulo fiberwise vector fields, which has a simple zero on the two fibers over the branch points of the period map.

V.3. General vector fields

Having understood fiberwise vector fields, and vector fields modulo fiberwise vector fields, there is the remaining question of lifting what we have called local transverse vector fields to actual local vector fields, or more generally, the extension question.

It is easiest to start with just the double cover, let's call it D, of $\mathbb{P}^2$ along $L_1 \cup L_2 \cup L_3 \cup L_4$. Calling the double covering map π we know that due to a nice and general property of logarithmic one-forms,

$$\Omega_D(log(L_1 \cup L_2 \cup L_3 \cup L_4) \cong \pi^*\Omega_{\mathbb{P}^2}(log(L_1 \cup L_2 \cup L_3 \cup L_4).$$

Here the L_i on the left side of the equation refer to the reduced divisors which support the pullback of the L_i as a divisor on $\mathbb{P}^2$.

As we may do with any branched cover, we can use the identification to explicitly construct one-forms on the cover. We construct the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \Omega_D & \to & \Omega_D(log(L_1 + ... + L_4)) & \to & \oplus_{i=1}^4 \mathcal{O}_{L_i} & \to & 0 \\ & & \uparrow & & \uparrow\cong & & \uparrow & & \\ 0 & \to & \pi^*\Omega_{\mathbb{P}^2} & \to & \pi^*\Omega_{\mathbb{P}^2}(log(L_1 + ... + L_4)) & \to & \oplus_{i=1}^4 \pi^*\mathcal{O}_{L_i} & \to & 0 \end{array}.$$

This shows that Ω_D is isomorphic to the kernel of the composite

$$\pi^*\Omega_{\mathbb{P}^2}(log(L_1 + ... + L_4) \to \oplus_{i=1}^4 \pi^*\mathcal{O}_{L_i} \to \oplus_{i=1}^4 \mathcal{O}_{L_i}.$$

We should now be able to analyze this very precisely. A crude first statement of the corresponding principle for vector fields is that the general holomorphic vector fields on $\mathbb{P}^2$ which preserve some subset of L_1, L_2, L_3, L_4 should lift to a meromorphic vector field on the double cover with simple pole on whichever of the L_i are not preserved in $\mathbb{P}^2$.

IV.4. A particular fiberwise vector field

Let's revisit our analysis of fiberwise vector fields using this principle to try to find an extension of it to the compactification. We start with a vector field on $\mathbb{P}^2$ which preserves the singular foliation by lines through $[0:0:1]$. This fixes that point and the line where $z=0$, and so the vector field lifted to the double cover and then to the surface where the inverse image of $[0:0:1]$ is two exceptional rational curves E_1, E_2, has as its locus of zeroes E_1 and E_2 together with the line L_4 and its locus of poles L_1, L_2, L_3. The divisor is $E_1+E_2-L_1-L_2-L_3+L_4$. Writing the scroll F_1 as the union of M_2 and M_{-2} glued along the complement of the zero sections, a section of M_2 as a rational function has its zeroes the zero section of M_{-2} and the fiber over the zero of that section of M_2, and as its poles the zero section of M_2. If we write h for the divisor class of a section of M_2 and e for the exceptional divisor class, the class of a section of M_{-2} and f the class of a fiber, we have on the scroll $0=e+f-h$ since it is the class of a principal divisor. Whereas the divisor of our fiberwise vector field on our surface is the pullback of $-e_h=f$. This is the divisor class of one fiber of our surface. So we have verified again that fiberwise vector fields on our surface with six nodes have divisor of poles equivalent to one fiber.

Let's try to understand this in a simpler way. The vector field $z\frac{\partial}{\partial z}$ on affine space induces a well-defined vector field on $\mathbb{P}^2$ and is equal to that induced by $-x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}$. The corresponding vector field on F_1 has a simple zero on the exceptional line and the line L_4 defined by the equation $z=0$. It lifts to the double cover having a simple zero on each of the two disjoint exceptional lines E_1, E_2, and just a simple zero on the L_4, but acquires simple poles on the lines L_1, L_2, L_3. The divisor $E_1+E_2-L_1-L_2-L_3+L_4$ restricts to a principal divisor on each elliptic curve fiber, but without restricting, it is equivalent to minus the class of a fiber.

By the usual conventions, one writes

$$g_2 = -4(e_1e_2 + e_2e_3 + e_1e_3)$$

$$g_3 = 4e_1e_2e_3.$$

We can multiply by $\sqrt{4 - \frac{g_2}{z^2} - \frac{g_3}{z^3}} = \sqrt{\frac{(z-e_1)(z-e_2)(z-e_3)}{(z-e_4)^3}}$ and the resulting vector field will have divisor $E_1 + E_2 - 2L_4$ and for numbers a, b if we multiply by $\frac{z}{ax+by}$ it will have divisor minus the inverse image of $[-b : a]$.

Let's look at the action of the vector field we're calling $-x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ on the affine part of $\mathbb{P}^2$ with coordinates $x/z, y/z$. It acts with eigenvalue -1 on each coordinate, so we would describe it in these coordinates as

$$-\frac{x}{z}\frac{\partial}{\partial \frac{x}{z}} - \frac{y}{z}\frac{\partial}{\partial \frac{y}{z}}.$$

When we multiply by our ratio $\frac{z}{ax+by}$ and then by $-\frac{1}{2}\sqrt{4 - \frac{g_2}{z^2} - \frac{g_3}{z^3}}$ we obtain

$$\frac{1}{2}\sqrt{4 - \frac{g_2}{z^2} - \frac{g_3}{z^3}}(\frac{x}{ax+by}\frac{\partial}{\partial \frac{x}{z}} + \frac{y}{ax+by}\frac{\partial}{\partial \frac{y}{z}}).$$

This lifts to a vector field on our elliptic surface with six nodes, having a simple pole on the fiber of one of the two connected irreducible components of the branching of the quadratic map induced by the period map.

For $\lambda = 1 - \frac{y}{x}$

$$1 - \gamma(\lambda) = 1 - \frac{1-\lambda^2}{1-2\lambda}$$

$$= \frac{\lambda^2 - 2\lambda}{1-2\lambda} = \frac{x^2 - y^2}{x^2 - 2xy}.$$

Define the rational quadratic differentials on $\mathbb{P}^1$ (recall x, y are rational one-forms on $\mathbb{P}^1$)

$$u = x^2 - 2xy$$

$$v = x^2 - y^2$$

so that

$$\gamma(\lambda) = 1 - \frac{v}{u}.$$

Let's take a moment to explain how to visualize this. By identifying any one of the lines L_1, L_2, L_3, L_4 in $\mathbb{P}^2$ with the Riemann sphere (they all correspond by moving along points of lines through $[0 : 0 : 1]$), we can interpret $\mathbb{P}^2$ as the smooth cone on that Riemann sphere, and we can choose an extension of the period mapping of the Riemann sphere extends to a quadratic map $\mathbb{P}^2 \to \mathbb{P}^2$. If we do this for the line L_4 then we identify L_4 which is the 'line at infinity' in $\mathbb{P}^2$, defined by $z = 0$ in coordinates $[x : y : z]$, with the projective line consisting of the ratios $[x : y : 0]$. This is the 'line at infinity' in the sense that $\mathbb{P}^2$ is the disjoint union of the affine plane which is the set of $[x : y : 1]$ with the line at infinity which is the set of $[x : y : 0]$. Extending the period map on L_2 to all of $\mathbb{P}^2$ and then restricting to the affine plane gives the two quadratic forms above as coordinates.

The points where L_1, L_2, or L_3 meet L_4 are the points $[0 : 1], [1 : 0]$, and $[1 : 1]$ of $L_4 = \mathbb{P}^1$, and the lines through $[0 : 0 : 1]$ which pass through an intersection point between two of L_1, L_2, L_3 pass through L_4 at $[1 : 2], [2 : 1]$ and $[-1, 1]$.

Taking (w, τ) as our coordinate function pair on $\mathbb{C} \times \mathbb{H}$ as usual, for each pair of numbers α, β not both zero, the vector field

$$\sqrt{\frac{d\tau}{\alpha u + \beta v}} \frac{\partial}{\partial w}$$

is invariant under the $\Gamma(2)$ action of weight $k = -1$ and descends to a vector field on the copy of M_{-1} which is a bundle over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, an open subset of the codomain of the quadratic period map.

The term under the square-root sign is merely a function with domain $\mathbb{H}$ because both numerator and denominator are one-forms. And the whole product is a $\Gamma(2)$ invariant vector field.

When in a later section we make a distinction between two ways we could use the variable τ, it will end up being appropriate to call the variable occuring above $\eta(\tau)$ in place of τ, as it is the variable relevant for the codomain of the period map; and on our elliptic surface S it is the actual period ratio of elliptic curves which double cover projective lines through $[0:0:1]$.

In a previous section we gave values

$$\alpha = e^{7\pi i/6}$$

$$\beta = e^{11\pi i/6}$$

so that the vector field on the double cover has a simple pole on the fiber over one of the the branched (and fixed) points of γ.

The coefficient is meant to be a weakly modular form of weight -1 for the group $\langle T^2, ST^2S\rangle \subset SL_2(\mathbb{Z})$.

On the branched cover induced by the quadratic period map, this pulls back to

$$\sqrt{\frac{d\tau}{\alpha(x^2-2xy)+\beta(x^2-y^2)}}\frac{\partial}{\partial w}$$

Somewhat magically, with the specified values of α, β the denominator under the square root sign is a perfect square, it is our positive real constant c times

$$-i(x-e^{-i\pi/3}y)^2$$

Therefore our invariant vector field is

$$\frac{e^{5i\pi/6}\sqrt{d\tau}}{(e^{i\pi/3}x-y)}\frac{\partial}{\partial w}.$$

Here, to say it again because this can be confusing: the expression $\sqrt{d\tau}$ refers to the variable τ which is a coordinate on the copy of $\mathbb{H}$ which covers the codomain $\mathbb{P}^1 \setminus \{0,1,\infty\}$ as is appropriate, because pulling back along the period map ought not affect period ratios. Whereas, x and y are sections of the line bundle of one-forms on the domain of the period map with at most simple (=logarithmic) poles at $0,1,\infty$ and poles are not allowed at the other points $[1:2],[2:1],[-1:1]$ which are their transforms under the order-two Galois automorphism.

That line bundle is isomorphic to $\mathcal{O}(1)$, and identifying sections of $\mathcal{O}(1)$ with sections of the structure sheaf of the surface M_{-2} which, as holomorphic functions, commute with the action of the multiplicative group on the line fibers, we may, away from the zero-section which becomes contracted under a map to the affine plane in $\mathbb{P}^2$, consider x and y to be affine coordinates in an affine plane.

Taking

$$a = e^{3i\pi/2}$$
$$b = e^{i\pi/6}$$

our invariant form is

$$\frac{\sqrt{d\tau}}{ax+by}\frac{\partial}{\partial w}.$$

It is a curious question whether we are allowed to identify this fiberwise vector field with the vector field with a simple pole at the point where $ax+by=0$ which we obtained before, which is the rational vector field on $\mathbb{P}^2$

$$\frac{1}{2}\frac{\sqrt{4-\frac{g_2}{z^2}-\frac{g_3}{z^3}}}{a\frac{x}{z}+b\frac{y}{z}}\left(\frac{x}{z}\frac{\partial}{\partial\frac{x}{z}}+\frac{y}{z}\frac{\partial}{\partial\frac{y}{z}}\right)$$

$$=\frac{1}{2z}\frac{\sqrt{4z^4-g_2z_2-g_3z}}{ax+by}\left(\frac{x}{z}\frac{\partial}{\partial\frac{x}{z}}+\frac{y}{z}\frac{\partial}{\partial\frac{y}{z}}\right).$$

The coefficient outside the parentheses is invariant under the multiplicative group action on (x, y, z) space, and the part in parentheses is the equals the well-defined descent to $\mathbb{P}^2$ of the action of the vector field $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ on affine space, which descends to a vector field on projective space, agreeing with the descent of $-z\frac{\partial}{\partial z}$.

In conclusion, we have the suggested equation

$$\frac{\sqrt{d\tau}}{ax+by}\frac{\partial}{\partial w} = \frac{1}{2z}\frac{\sqrt{4z^4 - g_2 z_2 - g_3 z}}{ax+by}(\frac{x}{z}\frac{\partial}{\partial\frac{x}{z}} + \frac{y}{z}\frac{\partial}{\partial\frac{y}{z}})$$

for

$$a = e^{3i\pi/2}$$
$$b = e^{i\pi/6}.$$

VI. Integral calculus on elliptic curves.

VI.1 Expression of $\wp(w,\tau)d\tau$ as a rational one-form

Consider M_{-1} as a line bundle on $\mathbb{P}^1 \setminus \{0,1,\infty\}$. The map

$$\mathbb{C} \times \mathbb{H} \to \mathbb{C}^2$$
$$(w,\tau) \mapsto (w\theta(0,\tau)^2, w\theta(0,1+\tau)^2)$$

contracts the zero section $0 \times H$ to a single point at the origin, and it is invariant for $\langle T^2, ST^2S\rangle$ so it descends to a map

$$M_{-1} \to \mathbb{C}^2$$

which contracts the zero section $\mathbb{P}^1 \setminus \{0,1,\infty\}$ to a single point.

There is a corresponding map which contracts the zero-section of the tensor square M_{-2}, induced by

$$\mathbb{C} \times \mathbb{H} \to \mathbb{C}^2$$
$$(w,\tau) \mapsto (w^2\theta(0,\tau)^4, w^2\theta(0,1+\tau)^4).$$

This map

$$M_{-2} \to \mathbb{C}^2$$

extends to a map whose domain is the full line bundle M_{-2} on $\mathbb{P}^1$.

Up to isomorphism of (locally free rank one) coherent sheaves, nothing changes if we append $d\tau$ to both coordinates, and then each coordinate is w^2 times a one-form on $\mathbb{P}^1$ with at most simple (=logarithmic) poles at $0, 1, \infty$. The factors $\theta(0,\tau)^4 d\tau$ and $\theta(0,1+\tau)^4 d\tau$ as sections of the dual line bundle to M_{-2} can be interpreted as functions $M_{-2} \to \mathbb{C}$.

We can compactify the image $\mathbb{C}^2$ by considering that $\mathbb{P}^2$ is nothing but M_2 and M_{-2} glued along the complement of their zero sections by the antipodal map on the fibers (to result in the scroll F_1), then with the zero-section of M_{-2} contracted to a point.

Another way of compactifying the image is to replace the factor w^2 by the periodic factor $\frac{\pi^2}{3\wp(w,\tau)}$. The meromorphic map

$$M_{-2} \to \mathbb{C}^2$$

$$(w,\tau) \mapsto (\frac{\pi^2\theta(0,\tau)^4}{3\wp(w,\tau)}, \frac{\pi^2\theta(0,\tau+1)^4}{3\wp(w,\tau)})$$

extends to

$$M_{-2} \to \mathbb{P}^2$$

$$(w,\tau) \mapsto [\frac{\pi^2}{3}\theta(0,\tau)^3 : \frac{\pi^2}{3}\theta(0,1+\tau)^4 : \wp(w,\tau)].$$

The restriction of M_{-2} to $\mathbb{P}^1 \setminus \{0,1,\infty\}$ is a non-Galois cover of

$$(\mathbb{Z}^2 \rtimes \langle T^2, ST^2S\rangle)\backslash(\mathbb{C}\times\mathbb{H})$$

and once the zero section is contracted, this embeds in the double cover of $\mathbb{P}^2$ branched over L_1, L_2, L_3, L_4 as the complement of the lift of $L_4 = V(z)$ union $V(-2x+y) \cup V(x-2y) \cup V(x+y)$.

Thus one component of the critical locus is deleted, and also three lines are deleted which are asymptotic to the other three components of the critical locus, as $L_1 = V(z+2x-y), L_2 = V(z-x-y)$, and $L_3 = V(z-x+2y)$.

We obtain a different copy of M_{-2} on $\mathbb{P}^1 \setminus \{0,1,\infty\}$ by pulling back M_{-1} along the period map

$$\gamma : \mathbb{P}^1 \to \mathbb{P}^1$$

$$[x:y] \mapsto [x^2-2xy : x^2-y^2].$$

If we identify $\mathbb{P}^1$ with the line L_4 the period map induces the map

$$\mathbb{P}^2 \to \mathbb{P}^2$$

$$[x:y:z] \mapsto [x^2-2xy : x^2-y^2 : z^2].$$

This has the property that in the double cover of $\mathbb{P}^2$ branched over L_1, L_2, L_3, L_4, the pairs of lines through $[0:0:1]$ whose inverse images in the double cover are isomorphic, become identified upon applying this map. We do not currently attempt to take any branched cover after applying this map, only to state that two lines whose inverse images are isomorphic as elliptic durves become identified under the map.

Although it did not matter in the previous section (since the coefficient was homogeneous of degree zero), here, let's change our definition of x and y not to be rational one-forms on $\mathbb{P}^1$ with at worst simple poles at $0, 1, \infty$, but, rather, we divide these by $d\tau$ obtaining instead

$$x = \frac{\pi^2}{3}\theta(0,\tau)^4$$

$$y = \frac{\pi^2}{3}\theta(0,1+\tau)^4.$$

We can still interpret x and y as sections of $\mathcal{O}(1)$ (the coherent sheaf structure is not affected by removing the symbol $d\tau$).

Now, as before, the classical lambda function is

$$\lambda = 1 - \frac{y}{x} = \frac{x-y}{x}.$$

From the formulas

$$d\ log\ \lambda = i\pi\theta(0,1+\tau)^4 d\tau$$

$$= \frac{3}{\pi} iyd\tau$$

and

$$d\ log\lambda = \frac{x}{x-y}(-d\frac{y}{x})$$

$$= \frac{y}{y-x}(\frac{x}{y}d(\frac{y}{x}))$$

$$= \frac{y}{y-x}(\frac{dy}{y} - \frac{dx}{x}),$$

eliminating $d\ log\lambda$ gives

$$d\tau = \frac{\pi}{3i}\frac{1}{xy(y-x)}(xdy - ydx).$$

$$= \frac{i\pi}{3}\frac{1}{xy(x-y)}(xdy - ydx).$$

The expression $xdy - ydx$ which we might more rigorously have written $x\nabla(y) - y\nabla(x)$ is a global section in the kernel of

$$\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{P}(\mathcal{O}_{\mathbb{P}^1}(1)) \to \mathcal{O}_{\mathbb{P}^1}(2)$$

and is therefore a global section of $\mathcal{O}_{\mathbb{P}^1}(2) \otimes \Omega_{\mathbb{P}^1}$, a copy of the trivial line bundle and therefore this is the uniquely determined global section up to a scalar multiple.

Multiplying by the coefficient converts it to a rational section of $\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \Omega_{\mathbb{P}^1}$. Therefore by multiplying by a linear form $ax + by$ we obtain a rational one-form.

Our equation is an identity that holds on $\mathbb{H}$ when x, y are interpreted as merely being modular forms.

In terms of our analytic parametrization, if we now instead write

$$x(w,\tau) = \frac{\pi^2}{3\wp(w,\tau)}\theta(0,\tau)^4$$

$$y(w,\tau) = \frac{\pi^2}{3\wp(w,\tau)}\theta(0,\tau)^4$$

then the same formula will instead describe $\wp(w,\tau)d\tau$. Let's record this

$$\wp(w,\tau)d\tau = \frac{i\pi}{3xy(x-y)}(xdy - ydx).$$

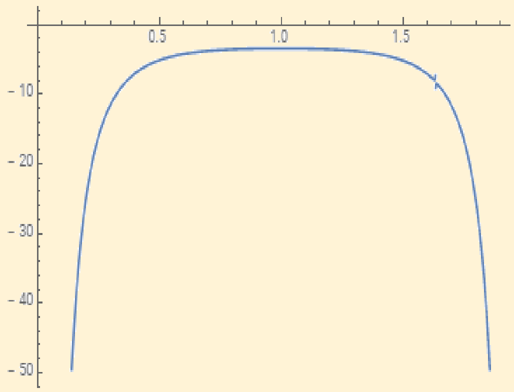
Let's state this

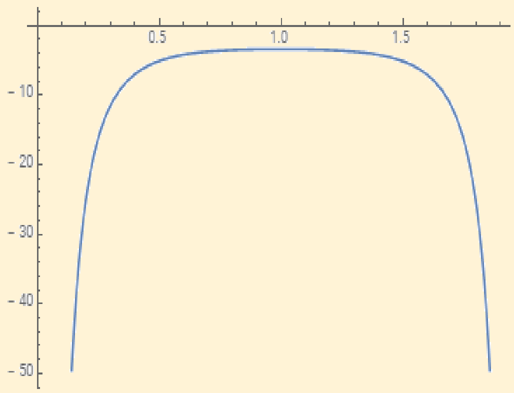
8. Theorem. The meromorphic one-form $\wp(w,\tau)d\tau$ on $\mathbb{C} \times \mathbb{H}$ is $\mathbb{Z}^2 \rtimes \langle T^2, ST^2S \rangle$-invariant for the action of degree -2. It is the pullback of the same rational one-form on $\mathbb{P}^2$ whose restriction to the affine (x, y) plane is

$$\frac{i\pi}{3xy(x-y)}(xdy - ydx).$$

To verify that this is not all a mistake, here is some Wolfram Alpha output to check one case using numerical differentiation. Note that Wolfram Alpha uses half-periods in the definition of $\wp$ and the third argument of EllipticTheta is q in place of τ.

```
X[w_, tau_] := Pi^2/3*EllipticTheta[3, 0, Exp[I*Pi*tau]]^4/WeierstrassP[w, WeierstrassInvariants[{tau/2, 1/2}]]
Y[w_, tau_] := Pi^2/3*EllipticTheta[3, 0, Exp[I*Pi*(1+tau)]]^4/WeierstrassP[w, WeierstrassInvariants[{tau/2, 1/2}]]
Test[w_, tau_] := I*Pi/(3*X[w, tau]*Y[w, tau]*(X[w, tau]-Y[w, tau]))*(X[w, tau]*ND[Y[w, t], {t, 1}, tau]-Y[w, tau]*ND[X[w, t], {t, 1}, tau])
tau := 2*I
Plot[Re[Test[1+t1*I, tau]], {t1, 0, 1.9}]
Plot[Re[WeierstrassP[1+t1*I, WeierstrassInvariants[{tau/2, 1/2}]]], {t1, 0, 1.9}]
```

Out[89]=


Out[90]=


Recall that by identifying $\mathbb{P}^1$ with L_4 and identifying $\mathbb{P}^2$ with the cone on L_4 with smooth cone point $[0 : 0 : 1]$, we extended the period map to a map

$$\mathbb{P}^2 \to V$$

where V is the quotient of $\mathbb{P}^2$ modulo the order-two automorphism

$$[x : y : z] \mapsto [x - 2y : 2x - y : -\sqrt{-3}z].$$

Here V a weighted projective plane of isomorphism type $\mathbb{P}(1, 2, 1)$ with one node at the image of $[0 : 1 : 0]$. If we compose this with the reduction of V modulo the automorphism induced by

$$[x : y : z] \mapsto [x : y : -z],$$

we can identify the image of the composite with $\mathbb{P}^2$ again, such that the composite is the map

$$\mathbb{P}^2 \mapsto \mathbb{P}^2$$
$$[x : y : z] \mapsto [x^2 - 2xy : x^2 - y^2 : z^2].$$

The Galois group of this map is a Klein four-group, generated by the automorphisms

$$\mathbb{P}^2 \to \mathbb{P}^2$$
$$[x : y : z] \mapsto [x - 2y : 2x - y : -\sqrt{-3}z].$$

and

$$[x : y : z] \mapsto [x : y : -z].$$

In new coordinates $[q : r : z]$ with

$$q = x + e^{2\pi i/6}y$$
$$r = x + e^{4\pi i/6}$$

the Galois group acts by negating q and r. It branches along the two lines defined by $x^2 - xy + y^2 = 0$ and the line defined by $z = 0$.

Under the map $\mathbb{P}^2 \to V$, the a singular variety isomorphic to weighted projective space where we take $[q : r^2 : z]$ as if they were homogeneous coordinates of degrees $1, 2, 1$. The projective lines through $[0 : 0 : 1]$ each map isomorphically, and except for the six special lines, and they become identified in pairs in the weighted projective plane upon reducing modulo the automorphism which negates r while fixing q and z.

When we reduce further by the full automorphism group the relevant ratios $[q^2 : r^2 : z^2]$ can be considered again to belong to a projective plane, but now the Riemann spheres which were the projective lines through $[0 : 0 : 1]$ map down with each experiencing a double cover branched at $[0 : 0 : 1]$ itself and with the 'point at infinity' where the line meets L_4 defined by $z = \infty$. We can still find all the elliptic curves, one each for the resulting projective lines, by taking a particular degree four cover instead of just the original degree two cover. We first take the double branched cover at $[0 : 0 : 1]$ and the meeting point with L_4 and then branch again at the meeting point with L_4 but also with the meeting points with L_1, L_2, L_3. The degree four cover is totally ramified at the meeting point with L_4.

Recall earlier we considered the holomorphic map which sends the six geodesic segments in order between these points to the three edges of the same triangle, so our map on vertices is

$$\begin{array}{ccc} i\infty & \mapsto & 0 \\ i & \mapsto & 1 \\ 0 & \mapsto & \infty \\ \frac{1+i}{2} & \mapsto & 0 \\ 1 & \mapsto & 1 \\ 1+i & \mapsto & \infty \end{array}.$$

This described a branched conformal map which we called η. It is a branched double cover of the ideal triangle, with the branch point of order two when $\tau = e^{2i\pi/6}$. It has the property that for our period map γ

$$\gamma(\lambda(\tau)) = \lambda(\eta(\tau)).$$

The map and the formula extend by symmetry to the second ideal triangle needed to cover $\mathbb{P}^1$ and the map has a second branch point in the second ideal triangle.

For each $\tau \in \mathbb{H}$ the line $\mathbb{C} \times \{\tau\}$ covers an elliptic curve which in turn covers a line through $[0 : 0 : 1]$ in $\mathbb{P}^2$. However, τ is not in general a period ratio for that elliptic curve. This is because we are not using Weierstrass' choice of branching (where $[0 : 0 : 1]$ would be a branch point for the elliptic curve double cover of interest to Weierstrass), but rather we are using the intersection of our line with the line $L_4 = V(z)$. That is why is $\eta(\tau)$ and not τ which is a period ratio for that elliptic curve.

For each τ, Let $\beta(\tau)$ be a reparametrization of the first variable so that

$$[x(\tau)^2 - 2x(\tau)y(\tau) : x(\tau)^2 - y(\tau)^2 : \wp(w,\tau)^2] = [x(\eta(\tau)) : y(\eta(\tau)) : \wp(\beta(\tau)(w), \eta(\tau)].$$

In other words, letting D be our fundamental domain, the union of two triangles, such that there is a commutative square

$$\begin{array}{ccc} \mathbb{C} \times D & \to & \mathbb{P}^2 \\ \downarrow & & \downarrow \\ \mathbb{C} \times D & \to & \mathbb{P}^2 \end{array}$$

where the leftmost downarrow is the function

$$(w, \tau) \mapsto (\beta(\tau)(w), \eta(w)).$$

By reducing modulo the lattice in each fiber in the copy of $\mathbb{C}\times\mathbb{H}$ on the upper right of the diagram and 'glueing' the boundary of both terms in the left column one can construct the complement of the singular fibers in the elliptic surface with six singular fibers; the map then contracts the zero section and what results is a branched cover of an open subset of $\mathbb{P}^2$. In the top row this can be completed to a branched cover of the whole of $\mathbb{P}^2$ which is a very algebraic object. If we reduce only modulo our order two automorphism there is an intermediate object, a weighted projective space, where all the lines through $[0:0:1]$ are intact but no longer distinct. We can resolve the singular cone point of this, now the map to $\mathbb{P}^1$ exists, and then after deleting the three lines $V(x), V(y), V(x-y)$ through $[0:0:1]$ what remains has a double cover which is a copy of the complement of the three singular fibers in the non-algebraic elliptic surface with three singular fibers for which the period map (the particular way we've defined it in the earlier sections of this paper) is an isomorphism.

The horizontal maps $\mathbb{C}\times\mathbb{H}\to\mathbb{P}^2$ in the commutative square are actually just two copies of the same map. That is, there is an endomorphism of that map which we could iterate as many times as we like, and at each stage, the the elliptic curve fibers have a period ratio which is given by the τ variable at the *next* stage.

It is interesting to pull back $\wp(w,\tau)$ through what are in this diagram the vertical maps one time. In our equation for $\wp(w,\tau)$, we modify the right side by putting x^2-2xy in place of x and x^2-y^2 in place of y. If $\beta(\tau)$ is invertible, on the right side of the equation we should replace the variables (w,τ) by $(\beta(\tau)^{-1}(w),\tau)$.

We haven't changed the letter τ, but we will change our interpretation of what it means, now it *does* refer to a period ratio for the fiber in the top row. Then after simplifying the expression

$$(x^2-2xy)d(x^2-y^2)-(x^2-y^2)d(x^2-2xy)$$

we have the multi-valued

$$\wp(\beta(\tau)^{-1}(w),\tau)d\tau = \frac{i\pi}{3}\frac{1}{(x^2-2xy)(x^2-y^2)(y^2-2xy)}((x^2-2xy)d(x^2-y^2)-(x^2-y^2)d(x^2-2xy)$$

$$= \frac{2i\pi}{3}\frac{x^2-xy+y^2}{x(x-2y)(x-y)(x+y)(y-2x)y}(xdy-ydx) \quad (3).$$

We consider this to be multivalued since for general τ we should not expect that the reparametrization $\beta(\tau)$ of w should have a single-valued inverse function.

The denominator in the coefficient is section of $\mathcal{O}(6)$ on $\mathbb{P}^1$ with zeroes at ratios $[x : y]$ where the corresponding line through $[0 : 0 : 1]$ meets one of the six intersection points among the lines L_1, L_2, L_3, L_4.

The numerator in the same coefficient is a section of $\mathcal{O}(2)$ with a zero at the two branching points of the period map γ.

The left side is now a $\mathbb{Z}^2 \rtimes (\langle T^2, ST^2S\rangle)$ invariant one-form on $\mathbb{C} \times \mathbb{H}$ which descends to a form on our elliptic surface. The right side is a rational one-form on affine space which extends uniquely to a rational one-form on $\mathbb{P}^2$. This pulls back to a rational one-form on our surface S with six singular fibers, which has a simple pole on each of the six singular fibers and a simple zero on the two smooth fibers where the map from S to the surface with just three singular fibers is branched. It also has a double pole on the two exceptional Riemann spheres which meet every smooth fiber once each, and a double zero on the inverse image in S of the line L_4, defined by the equation $z = 0$, which meets each elliptic curve once.

In our elliptic surface, each of the six singular fibers is the double cover of one of the six lines through $[0 : 0 : 1]$ which meets one of the six intersection points of L_1, L_2, L_3, L_4. Here, this covering copy of $\mathbb{P}^2$ contains six smooth rational curves meeting at $[0 : 0 : 1]$ which lie over the three rational curves which intersect the sphere at infinity at $0, 1, \infty$.

The four lines in this copy of $\mathbb{P}^2$ are such that the same six lines through $[0:0:1]$ here happen to be the ones which meet an intersection point among two of the lines corresponding isomorphically to L_1, L_2, L_3, L_4.

For either formula, note that the one-form $xdy - ydx$ has the property that for any rational function f of x and y,

$$df \wedge (xdy - ydx) = \delta(f)dx \wedge dy.$$

where δ is the derivation coming from the Euler derivation when we view x and y as sections of $\mathcal{O}_{\mathbb{P}^1}(1)$, namely

$$\delta = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

In the first variant, the coefficient is 'homogeneous of degree -3' and in the second, the coefficient is 'homogeneous of degree -6', and so $\delta(f) = -3f$ and $\delta(f) = -6f$ in the two cases.

In the first case, we finally get rid of that denominator of 3 and can interchange x and y to get rid of the sign, so we have the appealing formula

$$d\wp \wedge d\tau = \frac{i\pi}{xy(y-x)}dx \wedge dy.$$

Whereas, on the other side of both equations

$$d\wp(w,\tau) \wedge d\tau = (\frac{\partial}{\partial w}\wp(w,\tau))dw \wedge \tau.$$

And we know from Weierstrass' formula

$$\frac{\partial}{\partial w}\wp = 2\sqrt{(\wp - e_1)(\wp - e_2)(\wp - e_3)}.$$

(The factor of 2 is because we're choosing to use periods instead of half-periods in the definition).

At first sight this all looks surprising because on the right side of the equation the divisor is unaffected by applying the deRham differential. Any discrepancy must be hidden in the ratio between $dw \wedge d\tau$ versus $dx \wedge dy$. We know that the map does not have Jacobaian determinant constant; the Jacobian determinant is zero along the ramification locus.

VI.2 Variable of integration for $L(s, \chi, \tau)$.

We can use our formula for the $\Gamma(2)$ invariant one-form $\wp(w, \tau)d\tau$ to change the variable of integration in the formula for the modular L series into a rational variable, although the integrand will still involve τ. From

$$y = \frac{\pi^2\theta(0, 1+\tau)^4}{3\wp(w, \tau)}$$

we can rewrite

$$\wp(w, \tau)d\tau = \frac{\pi^2\theta(0, 1+\tau)^4}{3y}d\tau$$

and then

$$d\tau = \frac{3y}{\pi^2\theta(0, 1+\tau)^4}\frac{1}{xy(y-x)}(xdy - ydx)$$

Then from from $d\ log(\frac{\lambda}{q}) = i\pi(\theta(0, 1+\tau)^4 - 1)$

$$L(s, \chi, \tau) = -\Gamma(s)\pi^{1-s}\int_0^\tau (\frac{\tau}{i})^{1-s} i\pi(\theta(0, \tau)^4 - 1)d\tau$$

$$= -3(i\pi)^{-s}\Gamma(s)\int_0^\tau \tau^{1-s}(1 - \frac{1}{\theta(0, 1+\tau)^4})(\frac{1}{1-\frac{y}{x}})d\frac{y}{x}.$$

The upper limit of integration refers to the value of $\frac{y}{x}$ which is $1 - \lambda(\tau)$.

VI.3 Integration by substitution.

If $p(t)$ gives a path in a manifold, starting at time 0 (and we may consider time to be a complex holomorphic entity), and if a flow is given on the manifold, whose corresponding operator on functions is the derivation $\delta,$ then as long as the path p is an integral curve of the flow, parametrized accordingly, we may explicitly calculate for each complex number c

$$\int_0^c \delta(f)(p(t))dt = \int_0^c \frac{df}{dt}(p(t))dt$$

$$= \int_0^c df(p(t)) = f(p(c)) - f(p(0)).$$

And we can continue the analysis

$$= (e^{c\delta}(f) - f))(p(0)).$$

The element $c\delta$ belongs to the Lie algebra, and knowing how its exponential acts on f allows the calculation of the original integral of the real valued function of $t.$

A variant of this analysis is the following:

$$\int_0^c \frac{df(p(t))}{\delta(f)(p(t))} = \int_0^c \frac{\delta(f)(p(t))}{\delta(f)(p(t))}dt$$

$$= \int_0^c dt = c.$$

It is this second variant which explains why elliptic integrals calculate periods in the lattice in the Lie algebra. Starting with a one-form of the type $\frac{dz}{g(z)}$ one considers the derivation δ such that $\delta(z) = g(z).$ In other words, so that the function $g(z)$ is the contraction $i_\delta(dz)$ of the one-form dz which is in the numerator. Then one allows the path $p(t)$ to form itself starting from a point $p(0),$ using the exponential formula if one wishes to be explicit; and the integral of $\frac{dz}{g(z)}$ along this particular path has the property that the integral from $p(0)$ to $p(t)$ is just t itself for all values of $t.$

If $p(c) = p(0),$ the value of c is a period of $\frac{dz}{g(z)},$ and $(e^{c\delta}(g) - g)(p(0)) = 0.$

We can write the one-form related to Riemann's hypothesis as $\frac{d\frac{\lambda}{q}}{\frac{\lambda\tau^{1-s}}{q}}$

or, the one which is slightly more directly related, writing $u = -i\tau$, is $\frac{d\frac{\lambda}{q}}{\frac{\lambda u^{1-s}}{q}}$ and so we can try to choose δ such that

$$\delta(\frac{\lambda}{q}) = \frac{\lambda}{q}u^{1-s}.$$

Given any path $p(t)$ in the upper half plane, say, for t ranging from 0 to 1, we determine values of u along the path by solving the ordinary differential equation

$$\frac{d}{dt}u(p(t)) = \frac{\frac{d}{dt}p(t)}{u(p(t))^{s-1}i\pi(\theta(0, 1 + iu(p(t)))^4 - 1)}.$$

Then Writing $a = p(0)$ and $b = p(1)$, in terms of the modular L series

$$\frac{-\pi^{s-1}}{\Gamma(s)}(L(s, \chi, iu(b)) - L(s, \chi, iu(a)))$$

$$= \int_0^1 u(p(t))^{s-1}i\pi(\theta(0, 1 + iu(p(t)))^4 - 1)\frac{d}{dt}u(p(t))dt$$

$$= \int_0^1 \frac{d}{dt}p(t)dt = b - a.$$

Now choose instead an arbitrary point ie in the upper half plane (so e has positive real part). Choose a smooth path p with endpoints $p(0) = 0$, $p(1) = \frac{\pi^{s-1}}{\Gamma(s)}L(s, \chi, ie)$. Solve the differential equation above with $u(p(0)) = e$, and let $b = u(p(1))$. We have

$$\frac{-\pi^{s-1}}{\Gamma(s)}(L(s, \chi, ib) - L(s, \chi, ie)) = \frac{\pi^{s-1}}{\Gamma(s)}L(s, \chi, ie)$$

and therefore

$$L(s, \chi, ib) = 0.$$

The point $\tau = iu(b)$ is then a zero of $L(s, \chi, \tau)$. Note we do not assume u to be real. Thus

9. Observation. For each point ie in the upper half plane and each smooth real path p connecting 0 to

$$\frac{\pi^{s-1}}{\Gamma(s)} L(s, \chi, ie),$$

a solution u of the differential equation above with $u(p(0)) = e$, if it exists, satisfies that the number $\tau = iu(p(1))$ is a zero of $L(s, \chi, \tau)$.

10. Example. Take $s = .4 + .2i$, $e = 1$, and $p(t) = (1 - t)e + t\frac{\pi^{s-1}}{\Gamma(s)} L(s, \chi, ie) = (1-t) \cdot 1 + t \cdot (-.2057 + .1987i)$. Setting $u(p(0)) = u(0) = e$ and applying the differential equation gives $\tau = u(p(1)) = -.4142 + .1888i$, and for this value of τ we have $L(s, \chi, \tau) = 0$.

As we vary s we would see the position of τ changing.

11. Remark. The zeroes of $L(s, \chi, \tau)$ which tend to the ideal point $\tau = i\infty$ as s varies are those which correspond with zeroes of Riemann's zeta function.

Since we're restricting the value of s to have real part between zero and 1, the coefficient in the function

$$\mathbb{H} \to \mathbb{C}$$

$$\tau \mapsto \frac{-\pi^{1-s}}{\Gamma(s)} L(s, \chi, \tau)$$

is just a constant.

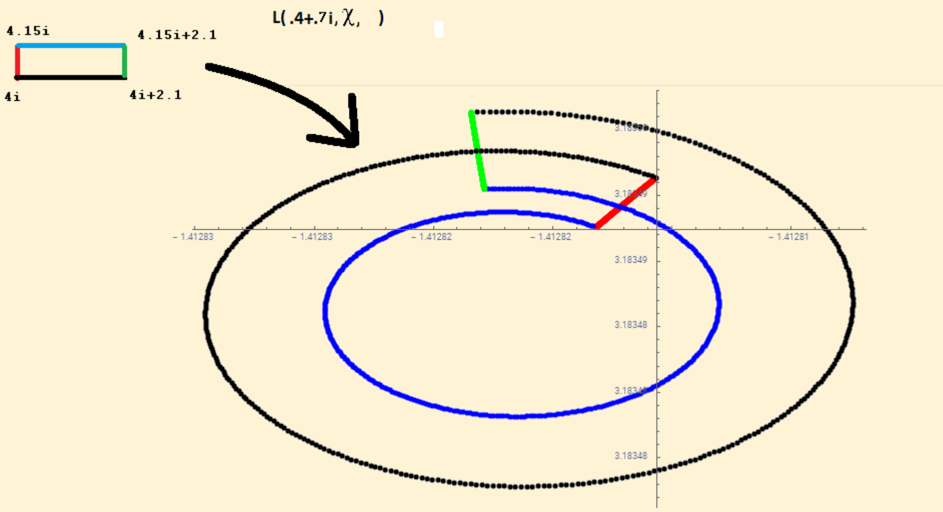
12. Observation. As a function of τ $L(s, \chi, \tau)$ is unbranched but not proper.

It is unbranched because $\theta(1/2, \tau) = \theta(0, 1 + \tau)$ does not take the value $0, 1, i, -i$ for any value of τ. Given this, then it is not proper because $\mathbb{H}$ is not conformally equivalent to $\mathbb{C}$.

VI.4. Deformation of T.

This opens the likelihood that it is almost certainly not not single-valued; i.e., not an embedding. The choice of path p of 'complex times' is essential choice leading to all the possible solutions of $L(s, \chi, \tau) = 0$ starting with any one initial nonzero value.

An example shows it is not single valued. Fix $s = .4 + .7i$, chosen at random; this small rectangle in the τ plane overlaps itself under the mapping. Lines of constant imaginary value tend to a circle near the black arc as the imaginary coordinate tends to zero, and to a single point, which is the value $L(s, \chi) = L(s, \chi, i\infty)$, as the imaginary coordinate tends to infinity.



This raises the question whether there is a neighbourhood U of the ideal point $i\infty$ in the upper half-plane, which is invariant under the action of an automorphism T_s which is a deformation of the automorphism $T : \tau \mapsto \tau + 2$ in the case when $s = 1$, such that for all $\tau \in U$ we have

$$L(s, \chi, \tau) = L(s, \chi, T_s(\tau)).$$

The specific way an infinite set of nearby solutions might arise, if Riemann's $\zeta(s) \neq 0$ but is near zero, for $0 < Re(s) < 1$, is that $L(s, \chi, i\infty) = L(s, \chi)$ would be near zero so the open set U should contain a zero of $L(s, \chi, \quad)$, and therefore that the infinite cyclic orbit of this zero under the action of T_s would produce an infinite set of zeroes. That is to say, we might define a holomorphic isomorphism η_s in a neighbourhood of the ideal point by the rule

$$e^{i\pi\eta_s(\tau)} = L(s, \chi, \tau) - L(s, \chi),$$

From invariance of the exponential map we'd have $\eta_s(T_s(\tau)) = \eta_s(\tau) + 2$. Remembering the ordinary transformation T such that $T(\tau) = \tau + 2$ we would have a formula for T_s as the conjugate

$$T_s = \eta_s^{-1} T \eta_s.$$

where

$$\eta_s(\tau) = \frac{1}{i\pi} \, (log(L(s, \chi, \tau) - L(s, \chi)).$$

As s approaches 1, our deformation η_s approaches the identity, and T_s approaches T itself.