JOURNAL OF ALGEBRA 122, 1–14 (1989)

Brauer Induction for G_0 of Certain Infinite Groups

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Communicated by A. W. Goldie

Received February 3, 1987

F. Quinn has recently proven a Brauer induction theorem for K_0 of certain infinite groups [9]. F. T. Farrell points out [3] that the theorem does not extend to more general coefficient rings. Arguments of Formanek have lead to such an extension on the level of trace functions, which requires an Artin exponent [5]. We begin by analyzing the structure of a Noetherian ring U graded by a virtually polycyclic group Γ, and with units in all degrees. For each such U and Γ and each finite $H \subset \Gamma$, denote by U_H the subring of U supported on H.

Then we prove surjectivity of the induction map:

THEOREM 1.

$$\bigoplus_{H \subset \Gamma} G_0(U_H) \twoheadrightarrow G_0(U_\Gamma). \tag{1}$$

The treatment of torsion is based on Farrell and Hsiang's idea (e.g., [3, 9]) of approximating the K-groups of a crystallographic group ring by a coefficient system of K-groups on a torus. For instance, in the special case that U is the group algebra over $\mathbb{Q}$ of a subgroup of finite index in $\mathbb{Z}^n \rtimes S_n$ acting in the obvious way on $\mathbb{R}^n$, the K-groups of the ring $R_v^{+\eta}[G]$ constructed in Section 3 will be sums of the vertex groups of such a coefficient system. Theorem 1 was announced in [5]. In case U is a group algebra over $\mathbb{Q}$, (1) follows from [9]. The proof here uses Quillen's [8] where our previous formulation, for trace functions, uses an elementary argument about graded rings.

The immediate consequences of (1) are two extensions of the well-known work of Brown and Farkas and Snider on zero-divisors [1, 2, 7], Rosset's conjecture [6], and the Goldie rank conjecture, in the unsolved prime characteristic case.

0021-8693/89 $3.00

[1]

[1] The article of F. Quinn referenced in [9] http://arxiv-web.arxiv.org/pdf/math/0509294v2.pdf has now appeared and solves the main issues which I was concerned about here.

The use of some negative K groups in calculating $K_0(\mathbb{Z}G)$ is described in his first announcement; this is consistent with a different notion, allowing H to be a possibly infinite hyperelementary group instead of a finite group, and for such groups H it is proven in Quinn's paper that

$$\oplus K_0(\mathbb{Z}H) \to K_0(\mathbb{Z}G)$$

is surjective.

In [4] these arguments will be used to extend the statement of the Goldie rank conjecture to solvable groups, and to prove the zero-divisor conjecture for solvable groups.

1. The Finite Subgroups of Γ

In the proof we may assume Γ is virtually abelian, as every polycyclic by finite Γ has a composition series with virtually abelian quotients.

Let us examine the structure of a particular type of virtually abelian group. Suppose Γ is virtually abelian and finitely generated. Write

$$0 \to M \to \Gamma \to G \to 1 \tag{2}$$

exactly where M is a finitely generated $\mathbb{Z}G$-module with no $\mathbb{Z}$-torsion and G is a finite group. Suppose that

$$M \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}[S], \qquad |S| < \infty, \tag{3}$$

where S is a free G-set. Equation (3) is the same as the condition that there exists an exact sequence

$$0 \to M \to \mathbb{Z}[S] \to E \to 0 \tag{4}$$

for some $\mathbb{Z}G$-module E with finitely many elements.

Then the long exact cohomology sequence of (4) gives an isomorphism

$$\cdots \to 0\, H^1(G, E) \xrightarrow{\delta_G} H^2(G, M) \longrightarrow 0 \cdots .$$

Say $\zeta \in H^2(G, M)$ is the class of the extension (4), and suppose

$$\zeta = \delta_G[e].$$

Here we are taking e to be a normalized cocycle.

Say $E = \{e_1, e_2, ..., e_n\}$. Define a G-action on $\{1, 2, ..., n\}$ by the formula

$$e_{g(i)} = {}^{g}e_i + e(g). \tag{5}$$

The cocycle condition

$$e(gh) = {}^{g}e(h) + e(g)$$

ensures that (5) defines a G-action. Conversely, one can reconstruct Γ from the module extension (4) and this G-action on (the subscripts of) the elements of $E=\{e_1, ..., e_n\}$. One simply uses (5) to define the cocycle e and sets

$$\zeta=\delta_G[e].$$

Our first observation is

LEMMA 2. *Let $H\subset G$. Then H is the image of a finite subgroup of Γ if and only if for some $i\in\{1, ..., n\}$*

$$H\subset G_i.$$

Proof. Now, H is the image of such a finite subgroup if and only if the class

$$[\mathrm{res}^G_H(-e)]=-\mathrm{res}^G_H[e]\in H^2(H, M)$$

is equal to 0. This is because

$$\delta_H\,\mathrm{res}^G_H[e]=\mathrm{res}^G_H\,\delta_G[e]=\mathrm{res}^G_H(\zeta),$$

where δ_H is the isomorphism in the long exact sequence

$$\cdots\to H^1(H, E)\to H^2(H, M)\to\cdots.$$

To say that $[\mathrm{res}^G_H(-e)]=0$ is the same as saying that for some $i\in\{1, ..., n\}$ there is an $e_i\in E$ such that

$$\begin{aligned}-e(g)&=\partial(e_i)(g)\\&={}^g e_i-e_i\qquad\text{for all}\quad g\in H.\end{aligned}$$

For this choice of i, this is equivalent to

$$e_{g(i)}={}^g e_i+e(g)=e_i,\qquad g\in H$$

or

$$H\subset G_i.\quad\blacksquare$$

The first application of formula (11) below will be to explicitly describe a subgroup $K_i\subset\Gamma$ mapping isomorphically to each $G_i\subset G$. Choose a

$$\beta\in Z^2(E, M)$$

symmetric and normalized such that

$$[\beta]\in H^2(E, M)$$

is the class of the extension (4). We then have

$$\mathbb{Z}[S] \simeq M \times^{\beta} E,$$

so $(m, e)(m', e') = (m + m' + \beta(e, e'), e + e')$, and we may define a derivation

$$r\colon G \to \mathrm{Funct}(E, M)$$

by

$$^{g}(m, e) = (^{g}m + r(g)(e), {}^{g}e).$$

r and β are related by

$$^{g}\beta(e, e') + r(g)(e + e') = \beta(^{g}e, {}^{g}e') + r(g)(e) + r(g)(e'). \tag{6}$$

Let us record the rules that describe the cocycle condition on β and the fact that r is a derivation

$$\begin{gathered} \beta(b, c) - \beta(a + b, c) + \beta(a, b + c) - \beta(a, b) = 0 \\ r(gh)(e) = {}^{g}r(h)(e) + r(g)(^{h}e). \end{gathered} \tag{7}$$

Recall also that β is normalized and symmetric. For

$$i, j \in \{1, ..., n\}, \qquad g \in G,\ m \in M$$

define

$$s(i, j, g) = -r(g)(e_j) + \beta(e_i, e(g) - e_i) - \beta(^{g}e_j, e(g) - e_i) \tag{8}$$

$$d(m, i, j) = (m - s(i, j, 1), e_j - e_i) \in M \times {}^{\beta}E \simeq \mathbb{Z}[S]. \tag{9}$$

Finally, write $\partial(\bar{e}) = \alpha$ where $\bar{e}$ is e followed by the inclusion $E \subset M \times {}^{\beta}E$, so $[\alpha] = \zeta$. One calculates

$$\begin{aligned} \alpha(g, g') &= {}^{g}e(g') - e(gg') + e(g) \\ &= r(g)(e(g')) + \beta(^{g}e(g'), e(g)) \in M. \end{aligned} \tag{10}$$

Some formal consequences (proofs omitted) of (5)–(10) are

$$s(^{gh}i, {}^{h}i, g) + {}^{g}s(^{h}i, i, h) = s(^{gh}i, i, gh) + \alpha(g, h) \tag{11}$$

$$s(i, j, 1) + s(j, k, 1) = s(i, k, 1) + \beta(e_k - e_j, e_j - e_i) \tag{12}$$

$$\begin{aligned} s(^{g}i, i, g) + {}^{g}s(j, {}^{g}j, g^{-1}) + \alpha(g, g^{-1}) - s(^{g}i, {}^{g}j, 1) \\ = -{}^{g}s(i, j, 1) + r(g)(e_j - e_i). \end{aligned} \tag{13}$$

For each $i \in \{1, ..., n\}$ let

$$K_i = \{(s({}^g i, i, g), g) : {}^g i = i\} \subset M \times_\iota^\alpha G \simeq \Gamma.$$[1]

Equation (11) shows that this a subgroup, and it clearly maps isomorphically onto G_i.

2. The Grading on $M_n U$

For each $(m, g) \in M \times_\iota^\alpha G = \Gamma$, let (m, g) also denote a choice of homogeneous unit of U of degree (m, g). Then

$$(m_1, g_1)(m_2, g_2) = \gamma((m_1, g_1), (m_2, g_2)) \cdot (m_1 + {}^{g_1}m_2 + \alpha(g_1, g_2), g_1 g_2) \tag{14}$$

for some unit $\gamma((m_1, g_1), (m_2, g_2)) \in U_1$. For $1 \leqslant i, j \leqslant n$ let $e_{ij} \in M_n U$ be the i, jth matrix unit.

For each $g \in G$, let

$$\psi(g) = \sum_{i=1}^{n} (s({}^g i, i, g), g) \cdot e_{g(i)i} \in M_n U. \tag{15}$$

By (11), for $g, g' \in G$

$$\psi(g)\,\psi(g') = \eta(g, g') \cdot \psi(gg')$$

for

$$\eta(g, g') = \sum_{i=1}^{n} \gamma((s({}^{gg'}i, {}^{g'}i, g), g), (s({}^{g'}i, i, g'), g'))\, e_{gg'(i)\,gg'(i)}.$$

It follows that each $\psi(g)$ is a unit, and the function

$$v \colon G \to \operatorname{Aut} M_n(U_M)$$

defined by

$${}^{v(g)}(x) = \psi(g) \cdot x \cdot \psi(g)^{-1} \tag{16}$$

descends to a homomorphism modulo conjugations by units.

By construction the $\psi(g)$ are linearly independent over $M_n(U_M)$, so the cocycle η and the function v describe a twisted group algebra

$$M_n(U_M)_v^\eta[G]$$

[1] In this notation α denotes the cocycle of (G, M) and $\iota \colon G \to \operatorname{Aut}_Z(M)$ describes the G-module structure on M, in the twisted cartesian product $M \times_\iota^\alpha G$.

isomorphic to $M_n(U)$. For $u \in U_M$ homogeneous of degree m and $1 \leqslant i, j \leqslant n$, define

$$\deg(ue_{ij}) = d(m, i, j).$$

THEOREM 3. *Writing $R = M_n(U_M)$, this rule defines a G-equivariant $\mathbb{Z}[S]$-grading on R.*

Proof. Equation (12) implies that

$$d(m, i, j) + d(w, j, k) = d(m + w, i, k)$$

so if $t \in U_M$ of degree w,

$$\deg(ue_{ij}) + \deg(te_{jk}) = \deg(ute_{ik}).$$

By (13), (15), and (16),

$$\deg^{v(g)}(ue_{ij}) = {}^g\deg(ue_{ij}) \qquad \text{for} \quad g \in G. \ \blacksquare$$

3. The Natural Isomorphism

For any sub-G-module $J \subset \mathbb{Z}[S]$ we may form the subring $R_J{}^{\eta}_{v}[G] \subset R^{\eta}_{v}[G]$.

We may suppose $\{1, ..., r\} \subset \{1, ..., n\}$ is a system of orbit representatives for the G-action. For $1 \leqslant j \leqslant r$ write

$$\eta_j = pr_j \circ \operatorname{res}^G_{G_j}(\eta)$$
$$v_j = pr_j \circ \operatorname{res}^G_{G_j}(v)$$

where $pr_j\colon R_0 \simeq U_0^n \to U_0$ is the jth projection. Each

$$U_0{}^{\eta_j}_{v_j}[G_j]$$

is embeddable in $R_0{}^{\eta}_{v}[G]$ under the map sending U_0 to the jth factor of R_0. Moreover, the map

$$u \cdot g \mapsto u \cdot \psi(g)$$

maps

$$U_0{}^{\eta_j}_{v_j}[G] \xrightarrow{\simeq} U_{K_j}.$$

Letting

$$y = (\underbrace{1, 1, ..., 1}_{r \text{ entries}}, 0, 0, ..., 0) \in R_0,$$

the isomorphisms above furnish $R_0{}^{\eta}_{v}[G]\cdot y$ and $y\cdot R_0{}^{\eta}_{v}[G]$ with the structure of right and left $U_{K_i}\times\cdots\times U_{K_r}$-modules, respectively, and the isomorphisms

$$\begin{aligned} R_0{}^{\eta}_{v}[G] &= R_0{}^{\eta}_{v}[G]\cdot y\cdot R_0{}^{\eta}_{v}[G] \\ &\simeq R_0{}^{\eta}_{v}[G]\; y\otimes_{U_{K_1}\times\cdots\times U_{K_r}} yR_0{}^{\eta}_{v}[G] \end{aligned}$$

and

$$y\cdot R_0{}^{\eta}_{v}[G]\cdot y=\prod_{j=1}^{r} U_0{}^{\eta}_{v_j}[G_j]\simeq U_{K_1}\times\cdots\times U_{K_r}$$

imply that the functors

$$(y\cdot -)\qquad\text{and}\qquad (R_0{}^{\eta}_{v}[G]\; y\otimes_{U_{K_1}\times\cdots\times U_{Kr}} -)$$

induce inverse equivalences of categories

$$R_0{}^{\eta}_{v}[G]-\text{mod}\rightleftarrows U_{K_1}\times\cdots\times U_{K_r}-\text{mod}.$$

Write

$$y_i=(0,\ldots,\underset{\substack{\uparrow\\ i\text{th place}}}{1},0,\ldots,0)\in R_0.$$

Under the identification $R^{\eta}_{v}[G]=M_nU$, we have

$$y_i=e_{ii}.$$

LEMMA 4. *The diagram below commutes up to a natural isomorphism.*

$$\begin{array}{ccc} R_0{}^{\eta}_{v}[G]-\text{mod} & \xrightarrow[\simeq]{(y_j\cdot -)_j} & \prod_{j=1}^{r}(U_{K_j}-\text{mod}) \\ \downarrow{\scriptstyle (R^{\eta}_{v}[G]\otimes_{R_0{}^{\eta}_{v}[G]}-)} & & \\ R^{\eta}_{v}[G]-\text{mod} & & \downarrow{\scriptstyle \oplus_{j=1}^{r}(U\otimes_{U_{K_j}}-)} \\ \| & & \\ M_nU-\text{mod} & \underset{(e_{11}\cdot -)}{\xrightarrow{\simeq}} & U-\text{mod} \end{array}$$

Proof. Let us study the functor

$$\bigoplus_{j=1}^{r} U\otimes_{U_{K_j}} y_j\cdot -.$$

Since for any U-module N, any j,

$$N\simeq e_{11}\cdot M_nU\cdot y_j\otimes_{U_{K_j}}N \text{ natural,}$$

we have

$$\begin{aligned}
&\bigoplus_{j=1}^{r} U \otimes_{U_{K_j}} y_j \cdot - \\
&\quad \simeq \bigoplus_{j=1}^{r} e_{11} \cdot M_r U \cdot y_j \otimes_{U_{K_j}} (y_j \cdot -) \\
&\quad \simeq \bigoplus_{j=1}^{r} e_{11} \cdot R_v^\eta[G] \cdot y_j \otimes_{U_{K_j}} (y_j \cdot -) \\
&\quad \simeq e_{11} \cdot R_v^\eta[G] \cdot y \otimes_{U_{K_1} \times \cdots \times U_{K_r}} (y \cdot -) \\
&\quad \simeq e_{11} \cdot R_v^\eta[G] \otimes_{R_{0}{}_v^\eta[G]} (R_0{}_v^\eta[G] \cdot y \otimes_{U_{K_1} \times \cdots \times U_{K_r}} y \cdot -).
\end{aligned}$$

and the part in parentheses is naturally isomorphic to the identity. ▮

4. The Resolution

For each element $s \in S \subset \mathbb{Z}[S]$ and each $i, j \in \{1, 2, ..., n\}$, there is an element of the form

$$ue_{ij} \in M_n U$$

of degree s, such that u is a homogeneous unit of U_M: One writes

$$s = (m, e) \in M \times {}^\beta E,$$

chooses j such that

$$e_j = e + e_i,$$

and chooses $u \in U$ to be a homogeneous unit of degree

$$s(i, j, 1) + m.$$

Then the definition of $d(, ,)$ shows that

$$d(m, i, j) = s.$$

Definition. An element x of R is *primitive* if

1. $x = ue_{ij}$ for a homogeneous unit of U_M,
2. $\deg(x) \in S$.

Definition. An element x of R is *degenerate* if 1 holds and $\deg(x) = 0$.

Let $R^+ = R_{\mathbb{N}[S]}$ be the part of R supported on $\mathbb{N}[S] \subset \mathbb{Z}[S]$.

THEOREM 5. *Let M be an arbitrary $R^+{}^\eta_v[G]$-module. Then M has a resolution by modules of the form*

$$R^+{}^\eta_v[G]\otimes_{R_0{}^\eta_v[G]}N,$$

which has length n.

Proof. For $j\geqslant 0$ let

$$C_j\subset \underbrace{R^+\otimes_{U_0}R^+\otimes\cdots\otimes_{U_0}R^+}_{(j+1)\text{-times}}\otimes_{U_0}M$$

be the U_0-submodule generated by the

$$x_0\otimes x_1\otimes\cdots\otimes x_{j+1}m,\qquad m\in M$$

satisfying 1–5 below. Here the U_0-module structures come from restriction along

$$U_0\subset M_n(U_0)\subset M_n(U_M)=R:$$

1. For $1\leqslant i\leqslant j$, x_i is a product of primitives.
2. x_0 is either degenerate or a product of primitives.
3. x_{j+1} is degenerate.
4. $x_1x_2\cdots x_j$ is a product of primitives of distinct degrees if $j\geqslant 1$.
5. $x_0x_1\cdots x_{j+1}\neq 0$.

Note that $R^\eta_v[G]$ acts on each C_j by

$$g\cdot x_0\otimes\cdots\otimes x_{j+1}m={}^{v(g)}x_0\otimes\cdots\otimes{}^{v(g)}x_{j+1}\cdot\psi(g)\cdot m$$

Property 4 implies that $C_{n+1}=0$.
For $1\leqslant j$ define

$$d_j\colon C_j\to C_{j-1}\qquad\text{and}\qquad a_j\colon C_{j-1}\to C_j$$

by

$$d_j(x_0\otimes\cdots\otimes x_{j+1}m)=\sum_{t=0}^{j}(-1)^t x_0\otimes\cdots\otimes x_tx_{t+1}\otimes\cdots\otimes x_{j+1}m$$

and

$$a_j(x_0\otimes\cdots\otimes x_jm)=\begin{cases}0, & x_0''\text{ degenerate}\\ x_0'\otimes x_0''\otimes x_1\otimes\cdots\otimes x_jm; & \text{otherwise,}\end{cases}$$

where x_0', x_0'' satisfy

$$x_0' x_0'' = x_0;$$

each is a product of primitives, and x_0'' is chosen of maximal degree such that

$$x_0'' x_1 x_2 \cdots x_j$$

is a product of primitives of distinct degrees. Also define

$$b_j \colon C_j \to C_{j-1}$$

by

$$\begin{aligned} b_j(x_0 \otimes x_1 \otimes \cdots \otimes x_{j+1} m) \\ = x_0 \otimes x_1 \otimes \cdots \otimes x_j x_{j+1} m, \end{aligned}$$

and define $c_j = (-1)^{j+1}$. One then has

$$db - bd = cbb$$

$$cb + bc = ca + ac = 0$$

$$ab - ba = c(1 - da - ad)$$

as maps $C \to C$, with the convention $a_0 = b_0 = d_0 = 0$. Since ab is locally nilpotent, $1 - cab$ is a unit. Let

$$H = (1 - cab)^{-1} a.$$

If there were any $x \in C$ with $x \neq dHx + Hdx$, homogeneous of degree $\geqslant 1$, then there would be such an x with

$$(1 - dH - Hd)\, bax = 0.$$

Since $H = a + cHba$,

$$\begin{aligned} dHx &= dax - cdHbax \\ &= dax - c(1 - Hd)\, bax \\ &= dax - cbax + c(1 - cab)^{-1}\, adbax. \end{aligned}$$

Using this identity, $(1 - cab)(1 - dH - Hd)x$ simplifies to

$$((1 - ad - da) - c(ab - ba))\, x + ca(cbb - db + bd)\, ax = 0 + 0 = 0,$$

contradicting the choice of x. Therefore no such x exists and $dH + Hd = 1$ in positive degree.

Therefore C is a resolution of M.

It remains to show that each C_j is induced from $R_0{}^\eta_v[G]$. This is the same as showing that the underlying R^+-module is induced from R_0, which is obvious because x_0 ranges freely over an R_0-module generating set of R^+. ▮

Corollary 6. *Any U-module M has a length n resolution by modules of the form*

$$\bigoplus_{j=1}^{r} U \otimes_{U_{K_j}} N_j$$

Proof. In view of Lemma 4, it suffices to show that any $R^\eta_v[G]$ module M has a length n resolution by modules of the form

$$R^\eta_v[G] \otimes_{R_0{}^\eta_v[G]} -.$$

Note that up to units of U_0, there is a unique homogeneous unit $w \in R$ such that

$$\deg(w) = \sum_{s \in S} s \in Z[S].$$

The inclusion $R^+{}^\eta_v[G] \subset R^\eta_v[G]$ can be viewed as the Ore localization at the multiplicatively closed subset generated by w and the units of $R^+{}^\eta_v[G]$. Therefore any such M takes the form

$$R^\eta_v[G] \otimes_{R^+{}^\eta_v[G]} M^+$$

for some module M^+. Applying Theorem 5 to M^+ and applying the functor $R^\eta_v[G] \otimes_{R^+{}^\eta_v[G]} -$ to the resulting resolution of M^+ yields the desired resolution of M. ▮

At this point, note that, for the purpose of proving Theorem 1, the hypothesis (4) on the $\mathbb{Z}G$-module M can be ignored. Thus, suppose Γ is instead an arbitrary finitely generated abelian group. One still has

$$0 \to M \to \Gamma \to G \to 1$$

for some finitely generated $\mathbb{Z}G$-module M with no $\mathbb{Z}$-torsion. Now, choose a finitely generated $\mathbb{Q}G$-module N' such that

$$(M \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus N' \cong \mathbb{Q}[S]$$

for some finite G-set S. Of course, this can be done since finitely generated

$\mathbb{Q}G$-modules are projective. Also, since $-\otimes_{\mathbb{Z}} \mathbb{Q}$ commutes with pullbacks, letting $N = N' \cap Z[S]$, we have

$$(M \oplus N) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}[S].$$

Letting Γ act on N via G, we have

$$0 \to M \oplus N \to N \rtimes \Gamma \to G—1.$$

Choose a basis $T_1, ..., T_r$ for N, and replace U with the ring $U' = 1$,

$$U' = U[T_1, T_1^{-1}, ..., T_r, T_r^{-1}],$$

where the T_i commute with U_1, and otherwise the group N of monomials in the T_i and T_i^{-1} is preserved under conjugation by all units of U. Moreover, the resulting Γ-action on N is to be taken to be the one agreeing with the given structure of N as a Γ-module.

One can now identify U with the subring $U'_\Gamma \subset U'_N \rtimes \Gamma$. The groups Γ and $N \rtimes \Gamma$ share the same finite subgroups, and for each such finite subgroup H, the inclusion of U'_H in U', followed by the retraction of U' onto U in which each T_i maps to 1, equals the inclusion of U_H in U.

Now, suppose we have proven Theorem 1 for groups such as $N \rtimes \Gamma$. Then we will have

$$\bigoplus_{H \subset \Gamma} G_0(U_H) = \bigoplus_{H \subset N \rtimes \Gamma} G_0(U'_H) \twoheadrightarrow G_0(U'),$$

and in view of the remarks above, the following lemma will imply that Theorem 1 holds as well for U.

LEMMA 7. *The commutative diagram*

$$\begin{array}{ccc} & U' & \\ {}^{i}\nearrow & & \searrow^{\phi} \\ U & \xrightarrow{=} & U \end{array}$$

induces a commutative diagram

$$\begin{array}{ccc} & G_0(U') & \\ {}^{i_*}\nearrow & & \searrow^{\phi_*} \\ G_0(U) & \xrightarrow{=} & G_0(U); \end{array}$$

in particular, the map $G_0(U') \to G_0(U)$ *is surjective.*

A proof of Lemma 7 is contained in a manuscript by Farkas and Linnell, currently in preprint form. Put simply, one sets $\phi_*[X] = \sum_i' (-1)^i [\mathrm{Tor}_i^{U'}(U, M)]$, and checks $\phi_* i_* = \text{identity}$.

Now, we would be done if the U_{K_j}-modules N_j supplied by Corollary 6 happened to be finitely generated. In general they are not, of course; however, if one chooses M to be an $R^+{}^{\eta}_{v}[G]$-module whose underlying $R_0{}^{\eta}_{v}[G]$-module happens to be projective, the construction of Theorem 5 yields a finite projective resolution of M. In particular $R_0{}^{\eta}_{v}[G]$ has finite Tor dimension over $R^+{}^{\eta}_{v}[G]$. As $R^+{}^{\eta}_{v}[G]$ is projective over $R_0{}^{\eta}_{v}[G]$ and Noetherian, after one grades

$$R^+{}^{\eta}_{v}[G]$$

by total degree Quillen's [8] Theorem 7 shows that the inclusion

$$R_0{}^{\eta}_{v}[G] \subset R^+{}^{\eta}_{v}[G]$$

in fact induces an isomorphism on G_0. As the inclusion $R^+{}^{\eta}_{v}[G] \subset R^{\eta}_{v}[G]$ is an Ore localization, it induces a surjective map on G_0. Therefore the composite $R_0{}^{\eta}_{v}[G] \subset R^{\eta}_{v}[G]$ induces a surjection. But Lemma 4 implies that

$$\begin{array}{ccc} G_0(R_0{}^{\eta}_{v}[G]) & \twoheadrightarrow & G_0(R^{\eta}_{v}[G]) \\ \downarrow \wr & & \downarrow \wr \\ \bigoplus_{j=1}^{r} G_0(U_{K_j}) & \longrightarrow & G_0(U) \end{array}$$

commutes.

Acknowledgments

This paper developed from an ongoing attempt to introduce torsion into Bass–Formanek's work on linear groups. I thank my thesis advisor Hyman Bass, who indicated the Key expository papers and deftly fielded many incorrect proofs. I again thank Tom Farrell for explaining the relevant piece of his conjecture with Hsiang to me, and, finally, I thank Peter Linnell and Dan Farkas, who carefully read through my thesis and discovered the crossed product formulation of the induction theorem, important for the algebraic applications and buried in the proof of a lemma. They have prepared an exposition of (1), including a better method of proof based on a "co-restriction formula" for crossed product algebras. Since this paper was prepared, Cliff and Weiss have sent a paper containing a similar improvement, which independently works out Lemma 4 and much of [4] as well.

The formulation for G_0 instead of K_0 was suggested by M. Lorenz, K. A. Brown, J. Howie and they have already proven that the cokernel of (1) is torsion in the group ring case.

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