Finite-Dimensional Representations of Artin's Braid Group.

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Dedicated to the memory of Wilhelm Magnus

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1 Some linear representations.

This paper attempts a brief survey of what is currently known about an aspect of the linear representation theory of Artin's braid group. We shall freely use standard notation from the theory of braid groups, see [16], [1] or other papers in this volume.

A few years ago a survey such as this would have been short. Only one really interesting linear representation was known, the so-called *Burau representation*, see [1]. What "interesting" should mean is open to interpretation, but it seems reasonable to focus on those representations which have infinite image in a non-trivial way. As an example of what we wish to exclude, recall that a theorem of Baumslag implies that braid groups are residually finite and so have many linear representations with finite image. Moreover, any linear representation ρ may be adjusted by defining $\rho^*(\sigma) = \rho(\sigma).t^{\alpha(\sigma)}$ where $\alpha : B_n \to \mathbf{Z}$ is the abelianisation map and t is an indeterminate. This representation has infinite image, but could hardly be regarded as more interesting than the finite image representation. This is not to say that the finite representations of the braid group are uninteresting, (indeed they are probably very interesting), but rather that the investigation of finite representations is not a direction which we wish to pursue here.

The Burau representation admits many definitions, each in its own way giving some insight. The definition which is usually regarded as classical comes via free differential calculus. Fix the generators $x_1, ...x_n$ for a free group of rank n, and define a derivation on the group ring $\mathbf{C}[F_n]$ as follows. On the elements of the free group we define the derivation inductively by:

(i) $\frac{\partial(x_j)}{\partial x_i} = \delta_{ij}$

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(ii) $\frac{\partial(x_j.w)}{\partial x_i} = \delta_{ij} + x_j.\frac{\partial(w)}{\partial x_i}$ *where* $w \in F_n$.

We extend the derivation to the whole group ring by linearity. Then we may define the matrix corresponding to the n-braid σ by:

$$\beta_n(\sigma) = (\alpha\frac{\partial(\sigma(x_j))}{\partial x_i})$$

Here α is the abelianisation map of the previous paragraph extended to the group ring by linearity. This is a special case of a construction due to Magnus, see [1]. One finds that this representation is actually reducible and splits into an irreducible representation of dimension $n-1$ and a one dimensional representation which is trivial. The $n-1$ dimensional representation is what is usually referred to as the *reduced Burau representation*, we shall denote it by β_n^r.

An alternative description of these representations comes from the consideration of covering spaces. If we set D_n to be the n-punctured disc, choosing the basepoint on the boundary, we may form the fundamental group $\pi_1(D_n)$ and identify this in the obvious way with F_n. There is a homomorphism $r : \pi_1(D_n) \to \mathbf{Z}$ defined by sending each of the generators to the generator t of $\mathbf{Z}$. This homomorphism defines a covering $p : \tilde{D_n} \to D_n$. We consider the homology group $H_1(\tilde{D_n})$ as a module over $\Lambda = \mathbf{Z}[t, t^{-1}]$ where it becomes finitely generated and free of rank $n-1$. As is well known, there is a description of the braid group as orientation preserving homeomorphisms of D_n where two such homeomorphisms are regarded as equivalent if they are isotopic relative to the boundary. Given this description, we see that the group B_n acts as on $\tilde{D_n}$ and hence on the group $H_1(\tilde{D_n})$. One computes easily that this action is via module homomorphisms and we obtain a representation of the braid group into $GL(n-1, \Lambda)$; this is the reduced Burau representation.

For later use we also observe that one may also obtain the unreduced representation by consideration of relative homology groups. This is done in the following way. Let p_0 be the basepoint in D_n and let $\tilde{p_0}$ denote the full preimage of this point in $\tilde{D_n}$. Then the (unreduced) Burau representation arises as the action of the braid group on $H_1(\tilde{D_n}, \tilde{p_0})$; this is a Λ-module which is free of rank n. The natural sequence $H_1(\tilde{D_n}) \to H_1(\tilde{D_n}, \tilde{p_0}) \to H_1(\tilde{p_0})$ gives the splitting of Burau into trivial and unreduced parts. This point of view is clearly reminiscent of the covering space description of the Alexander polynomial of a knot or link. The fact that there is a concrete connection is between the Burau representation and the Alexander polynomial is well established: If α is a braid whose closure in the 3-sphere is the knot $\hat{\alpha}$, then apart from a normalisation factor, the Alexander polynomial of $\hat{\alpha}$ is given by $det(\beta_n^r(\alpha) - Id)$.

A good mathematical idea usually has many different interpretations, and yet another way of looking at the Burau representation is given in [13]. As is well known, the braid group B_n is a subgroup of $Aut(F_n)$. Starting with this fact, let $R = R(F_n, SU(2,\mathbf{C}))$ be the representation variety of F_n, topologized by the compact open topology. Fix once and for all some generating set $x_1, x_2, \ldots, x_n$ for F_n. Using this basis we see that since F_n is free any representation determines an n-tuple of matrices and any n-tuple of matrices determines a representation. Since the group $SU(2,\mathbf{C})$ is homeomorphic to the 3-sphere S^3, we may thus identify the representation space R with with $\mathcal{R} = S^3 \times S^3 \times \ldots \times S^3$. The action of an element of B_n as an automorphism of F_n then induces a diffeomorphism of $\mathcal{R}$, and the natural map $B_n \to Diff(\mathcal{R})$ is an injective group homomorphism. It turns out that there is a circle of fixed points. Parametrizing the circle by $e^{2\pi it}$, one finds that the induced action on the tangent space to $\mathcal{R}$ at a fixed point gives a linear representation of B_n, and since there is a one-parameter family of fixed points one obtains in this way a representation of B_n which contains the Burau representation.

A whole new family of representations was discovered by Jones in [9]. The construction given is much more mysterious and comes from the theory of Hecke algebras. (See [3]). One considers the **C**-algebra with a 1 which is generated by $g_1, ..., g_{n-1}$ and has relators:

(i) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ $1 \leq i \leq n-1$
(ii) $g_i g_j = g_j g_i$ $|i-j| > 1$
(iii) $g_i^2 = (q-1)g_i + q$

Denote this algebra $H_n(q)$; one way to view this algebra is as a deformation of the complex group algebra of the symmetric group Σ_n, which occurs in this setting as $H_n(1)$. We can summarise most of the salient propeties in the following theorem, the first proof of which is essentially due to Tits [3]:

Theorem 1.1 *(i) The algebra $H_n(q)$ has complex dimension $= n!$ for generic q.*
(ii) For q sufficiently close to 1, $H_n(q)$ is semisimple.
(iii) The simple $H_n(q)$ modules are in one to one correspondence with Young diagrams and their decomposition rules and dimensions are the same as for Σ_n.

We may define the *Jones representation* of B_n by mapping $\sigma_i \to g_i$ and then using the left regular representation. The theorem implies that the Jones representation is completely reducible and that the irreducible subrepresentations correspond to the Young diagrams for the representations of the symmetric group, Σ_n. It is convenient to use the terminology of Young diagrams, so we recall this briefly: The ordinary irreducible representations of the symmetric group Σ_n are parametrised by sequences of integers $n_1 \geq n_2 \geq .... \geq n_k$ with $\Sigma n_i = n$. Such a sequence is a *Young diagram* and will be annotated $(n_1, ..., n_k)$ with the convention that m^b is the sequence consisting of b consecutive appearances of m. Having (arbitrarily) decided which of (1^n) and (n) is the trivial representation and which corresponds to the signature homomorphism, then the diagram determines the representation. The only property we will use is the restriction rule which describes how the representation of Σ_n with Young diagram $\mathcal{Y}$ breaks up when considered as a representation of Σ_{n-1}; here the rule is the most natural one for which one might hope. Consider all possible Young diagrams obtained from $\mathcal{Y}$ by decreasing one of the n_j's by 1; this describes, with multiplicities, the representation of Σ_{n-1}. In principle this gives an inductive description of the representation corresponding to any Young diagram (given the convention above) although it is not very practical and direct methods exist.

The Jones representation is a generalization of the Burau representation in the sense that one of its irreducible summands is reduced Burau; we shall choose things so that it corresponds to Young diagram $(n-1,1)$. There are two simple ways to pick out summands which generalise the Burau summand. One is to consider the exterior powers. This is classical and is essentially what controls the Alexander module. The other collection of summands is all those of the shape $(n-m,m)$ where $m \leq n/2$. This is the *Temperley-Lieb algebra* and is the source of the original one-variable Jones polynomial, which appears as a normalised weighted trace function on this algebra. The two-variable polynomial is constructed as a certain trace on the whole Jones representation.

In [2] this process is actually reversed, and starting from the Kauffman polynomial of a knot, an algebra $\mathcal{C}_n(l,m)$ is constructed which yields another family of braid group representations, this time with two parameters. The algebra $\mathcal{C}_n(l,m)$ has dimension 1.3.5......$(2n-1)$, is semisimple and has quotients isomorphic to $H_n(q)$.

The representations of B_n in $H_n(q)$ and $\mathcal{C}_n(l,m)$ are but two special cases of finite-dimensional matrix representations of B_n which support a "Markov trace", and so give rise to polynomial invariants of knots and links. The description of the trace functions and the associated link invariants goes beyond the scope of this review, however it seems appropriate to describe the "method of R-matrices" which constructs them all. Let E be the ring of Laurent polynomials over the integers in a single variable $\sqrt{q}$, let $m \geq 1$ be an integer, and let V be a free E-module of rank m. For each $n \geq 1$ let $V^{\otimes n}$ denote the n-fold tensor product $V \otimes_E \ldots \otimes_E V$. Choose a basis $v_1, \ldots, v_m$ for V, and choose a corresponding basis $\{v_{i_1} \otimes \ldots \otimes v_{i_n}\}; 1 \leq i_1, \ldots, i_n \leq m\}$ for $V^{\otimes n}$. An E-linear isomorphism f of $V^{\otimes n}$ may then be represented by an m^n-dimensional matrix $(f^{j_1 \ldots j_n}_{i_1 \ldots i_n})$ over E, where the i_k's (resp. j_k's) are row (resp. column) indices.

The family of representations of B_n which we wish to describe have a very special form. They are completely determined by the choice of the integer m and an E-linear isomorphism $\mathrm{R} : \mathrm{V}^{\otimes 2} \to \mathrm{V}^{\otimes 2}$ (the so-called *R-matrix*) with matrix $[\mathrm{R}^{\mathrm{j_1 j_1}}_{\mathrm{i_1 i_2}}]$ as above. Let I_V denote the identity map on the vector space V. The representation $\rho(R) : B_n \to GL_{m^n}(E)$ which we seek is defined by

$$\rho(R) : \sigma_i \to I_V \otimes \ldots \otimes I_V \otimes \mathrm{R} \otimes \mathrm{I_V} \otimes \ldots \otimes \mathrm{I_V}$$

where R acts on the i^{th} and $(i+1)^{st}$ copies of V in $V^{\otimes n}$. Thus, if we know how R acts on $V^{\otimes 2}$ we know $\rho(R)$ for every natural number n.

What properties must R satisfy for $\rho(R)$ to be a representation? The first thing to notice is that if $|i-j| \geq 2$, the non-trivial parts of $\rho(\sigma_i)$ and $\rho(\sigma_j)$ will not interfere with one-another, so the first braid relation $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| \geq 2$ is satisfied by construction, independently of the choice of R. As for the second braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, it is clear that we only need to look at the actions of $\mathrm{R} \otimes \mathrm{I_V}$ and $I_V \otimes \mathrm{R}$ on $V^{\otimes 3}$. If

$$(\mathrm{R} \otimes \mathrm{I_V})(\mathrm{I_V} \otimes \mathrm{R})(\mathrm{R} \otimes \mathrm{I_V}) = (\mathrm{I_V} \otimes \mathrm{R})(\mathrm{R} \otimes \mathrm{I_V})(\mathrm{I_V} \otimes \mathrm{R})$$

then the second braid relation, and therefore both braid relations, will be satisfied. This equation is the clue to the construction. It is known as the *quantum*

Yang-Baxter (QYB) equation. It may be thought of as a combinatorial restriction on the entries in the matrix R. It turns out that the theory of quantum groups has lead to an effective classification of solutions to the QYB equation, and so to the construction of all possible R-matrix representations of B_n. For more on this subject, and for explicit examples, see [21] and [8].

2 Linearity and Effectiveness.

Given that linear representations exist, a natural question is whether the group B_n is actually a *linear group* that is to say, can it be *faithfully* represented as a group of matrices. The work of several authors, notably McCarthy [18] and Ivanov [7] has shown that many properties which a linear group must have are shared by braid groups. It is also known that if a faithful representation exists, there is an irreducible faithful representation. (See [5] or [12])

For $n = 2, 3$ it is known that the groups B_n are linear, the first case being a triviality and the second case proved by Magnus and Peluso in [17], where it is shown that the Burau representation is faithful for $n = 3$. To date all other cases of this question remain open. The case $n = 4$ is especially intriguing as it was shown in [5] that the linearity of B_4 is equivalent to the linearity of the group $Aut(F_2)$. Even more, Formanek and Processi have proved in ([6] that this is the only possible case when $Aut(F_n)$ could be linear. Structurally this group is simpler than the other braid groups and this means that special reductions are possible in this case. For example, B_4 contains a normal subgroup which is free of rank 2, and so it follows from [12] that a linear representation of B_4 is faithful if and only if the image of this free group is free of rank 2. This is a generalisation of the famous pair of matrices contained in [1] the freeness of which is shown to give a necessary and sufficient condition for the faithfulness of the Burau representation of B_4.

One can be less ambitious and ask: "When is the Burau representation faithful" and until 1990, this remained open despite work by many authors. Almost all cases are now covered by:

Theorem 2.1 ([19] & [15]) *The Burau representation is not faithful for* $n \geq 6$.

The is the combination of two results, the inital breakthrough of [19], where a slightly different point of view of the covering space description of the Burau representation was employed to show that β_k is not faithful for $n \geq 10$, together with a sharpening of this result in [15], which was used to show that β_k is not faithful for $n \geq 6$. The cases $n = 4, 5$ remain unresolved. This is despite the fact that [15] gives a criterion which is necessary and sufficient to determine faithfulness. We briefly describe the method and then recast the idea in a manner so that it can be generalised. Let ξ_j be the arc shown in Figure 1.

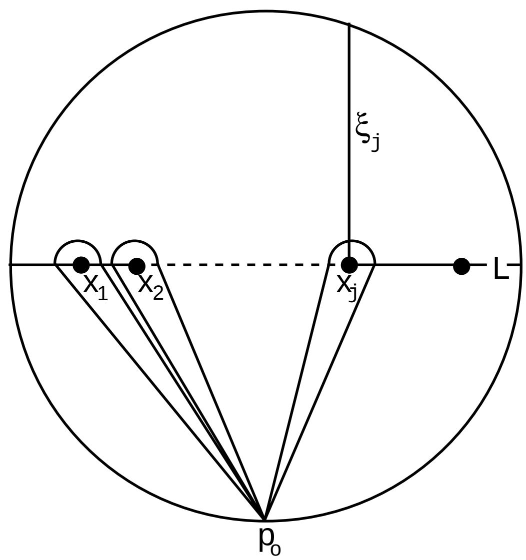

Figure 1

Then we define a map

$$\int \omega_j : H_1(\tilde{D}_n, \tilde{p_0}) \to \Lambda$$

by requiring that if α represents some homology class in $H_1(\tilde{D}_n, \tilde{p_0})$, we set

$$\int_\alpha \omega_j = \sum_{k \in Z} t^k.(\alpha, t^k \xi_j)$$

where $(\alpha, t^k\xi_j)$ is the algebraic intersection number of the arcs in $\tilde{D}_n$. With suitable conventions we may extend $\int \omega_j$ to D_n and one can compute the action of the Burau matrix of a braid by consideration of these integrals. Roughly speaking, $\int_\alpha \omega_j$ can be considered as an obstruction to isotoping the arc α off the arc ξ_j. The results of 2.1 are formulated in a way which gives elements in the kernel of the Burau representation if there are simple arcs for which certain of these obstructions vanish. The disadvantage with this method is that as it stands it is uniquely suited to the geometric description given for the Burau representation and to provide insight into representations such as the Jones representation it is necessary to have some more geometric description of them.

We now describe a construction which generalises the above and produces new representations of the braid group. It uses an idea which occurs both in the work of Magnus and in the work of Lawrence [11]: given a representation of the free group F_n and certain compatibility conditions one may construct a representation of B_n. The idea seems to be general enough to construct all the summands in the Temperley-Lieb algebra. Conjecturally it could construct all linear representations of the groups B_n, but this remains open. It is studied in detail in [14]. In order to describe the idea behind the construction of [14] we recall the notion of homology or cohomology of a space with coefficients in a flat vector bundle. Suppose that X is a manifold and that we are given a representation $\rho : \pi_1(X) \to GL(V)$. This enables us to define a flat vector bundle E_ρ: Let $\tilde{X}$ be the universal covering of X. The group $\pi_1(X)$ acts on $\tilde{X} \times V$ by $g.(\tilde{x}, \mathbf{v}) = (g.\tilde{x}, \rho(g).\mathbf{v})$. Then E_ρ is the quotient of $\tilde{X} \times V$ by this action. We now form the cohomology groups of 1-forms with coefficients in E_ρ, denoting these by $H^1(X;\rho)$ or $H^1_c(X;\rho)$ for compactly supported cochains. Relative versions also exist, but we shall omit discussion here. In order to get an action of the braid groups, recall that we have a natural inclusion of B_n as a subgroup of $Aut(F_n)$ so that there is a canonical way of forming a split extension $F_n \rtimes B_n$. It turns out that in order to get an action on the twisted cohomology group what is required is exactly a representation of this split extension:

Theorem 2.2 *Given a representation $\rho : F_n \rtimes B_n \to GL(V)$ we may construct another representation $\rho^+_s : B_n \to H^1_c(D_n;\rho)$ where s is another parameter.*

This works in exactly the way one might expect. The representation restricted to the free factor gives rise to the local system on the punctured disc and thus the twisted cohomology group and the compatibility condition provided by the split extension structure gives the braid group action.

Various comments are in order concerning Theorem 2.2. The first is that although the theorem is stated abstractly, there is a concrete recipe which enables one to write down the description of ρ^+_s given ρ. The second is that at first sight, it might seem that this theorem is of limited usefulness, since it requires a representation of the more complicated group, namely the split extension, but in fact the algebraic structure of the braid group is very well-understood, and so it has been known for some time that the group B_{n+1} contains subgroups isomorphic to $F_n \rtimes B_n$. Thus we deduce:

Theorem 2.3 *Given a representation $\rho : B_{n+1} \to GL(V)$, we may construct a representation $\rho^+_s : B_n \to H^1_c(D_n;\rho)$.*

The theorem shows that given a k parameter representation of the braid group, the construction yields a $k+1$ parameter representation, apparently in a nontrivial way. For example, if one starts with a (zero parameter) trivial representation of $F_n \rtimes B_n$, the theorem produces the Burau representation. However the role of this extra parameter is not purely to add extra complication - it also adds extra structure. For there is a natural notion of what it should mean for a representation of a braid group to be unitary (See [20], for example) and the results of Deligne-Mostow and Kohno imply:

Theorem 2.4 *In the above notation, if ρ is unitary, then for generic values of s, so is ρ^+_s.*

Moreover, we may iterate this construction and it is possible to identify a certain sequence of local systems with the construction of [11], where it is proved that such a procedure suffices to produce, as composition factors, the simple algebras in the Jones representation corresponding to the Temperley-Lieb algebra. Thus we have:

Theorem 2.5 [11] *Iteration of the augmenting construction, beginning with the trivial representation will eventually yield all summands of the Temperley-Lieb algebra.*

Our approach has the obvious advantage that it is extremely geometric and one can write down criteria for faithfulness or otherwise of the representations so produced in terms of cup and cap products in the twisted cohomology groups; this is the promised generalisation of the cohomology classes $\int \omega_j$. It leads to the notion of an *effective local system.* Roughly, effectiveness amounts to asking whether geometric intersections are detected by the algebra coming from Poincaré duality in the local system. If the pairing is effective this easily implies that the representation is faithful. The converse need not be true, however and in general it seems possible that the local system and braid group representation provided by $\rho : F_n \rtimes B_n \to GL(V)$ could be (respectively) noneffective and nonfaithful, but piece together to give a faithful representation ρ_s^+. However it is shown in [14] that this can happen only finitely often. The exact result requires the following notation.

Suppose that $\rho_n : F_n \rtimes B_n \to GL(V_n)$ is a sequence of representations with the property that $V_1 \subset V_2 \subset \dots$ and that if ρ_n is restricted to B_{n-1} this is the representation ρ_{n-1}. If all the ρ_n are faithful representations of B_n, there is nothing more to do so we suppose that this is not the case and we set r to be the smallest number for which ρ_r is not faithful when restricted to the braid group factor. We take s to be the smallest number so that the local system coming from ρ_s restricted to the free factor has noneffective intersection pairing. Finally, we define r^+ to be the largest number so that $\rho_{r^+}^+$ is a faithful representation of B_{r^+-1}. The result is:

Theorem 2.6

$$s \leq r^+ \leq s + 2r - 2$$

For example, consider the sequence of one dimensional representations τ_n : $F_n \rtimes B_n \to GL(\mathbf{C})$ where τ_n is induced from the representation of B_{n+1} where the generator σ_i is multiplication by the complex number t. In this case, $r = 3$ and although the exact value of s is not known, the results of [19] and [15] imply that $3 < s \leq 6$. The representation augments to Burau, and we deduce from 2.6 that the Burau representation has range of faithfulness given by $s \leq r^+ \leq s + 4$. In this case the information is weaker than what is already known.

In the case when $n \geq 4$ establishing whether a local system is effective or not seems to be a hard problem. Indeed, it seems possible that there are no effective local systems at all. Currently, even knowing this was true does not seem to suffice to show that the Temperley-Lieb representation of the braid group is eventually non-faithful; what this would show would be that for each fixed m, the representations $(n - m, m)$ become nonfaithful as n tends to infinity.

This problem highlights one of the difficulties of dealing "locally" with objects of the nature of the Jones representation which could be essentially global. For as pointed out above, by Theorem 2.2 of [12] one cannot make a faithful representation of B_n by piecing together sums of nonfaithful representations. This means that if one wants to deal with faithfulness questions, a summand by summand examination of say, the Jones representation is possible. However unless the information obtained is very detailed, this may not suffice to establish faithfulness of the whole representation. For there seems to be no particular reason for thinking that the Burau representation is alone in being "stably nonfaithful" amongst the summands of the Jones representation. But as n becomes large, the number of "types" of summands increases very rapidly, so that every time a particular type of summand becomes nonfaithful, some other type present is still sufficiently complicated so as to be faithful. Even to prove results about the Temperley-Lieb algebra, it would be necessary to sharpen Theorem 2.5 to show that eventually one obtains a representation of the braid group which contains all the two row representations simultaneously.

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