Riemann Roch following Fulton

Let $M \subset \mathbb{P}^N$ be an n dimensional smooth complex projective variety. For each locally free coherent sheaf $\mathcal{F}$ on M let $t(\mathcal{F})$ be the element of the Chow ring of M corresponding to the formal product of $\alpha/(1-e^{-\alpha})$ over the Chern roots α. Note if N is locally free with dual sheaf $\mathcal{N}$ then $t(N)ch(\Lambda^{\cdot}\mathcal{N}) = c_n(N)$ where odd terms in $\Lambda^{\cdot}\mathcal{N}$ count negatively; but we can't 'solve for t(N)' working only in the Chow ring of M. We abbreviate $t(\mathcal{T}_M)$ by $t(M)$.

We cannot exponentiate the normal bundle; instead embed $\mathbb{P}^N$ in $\mathbb{P}^N \times \mathbb{C}$ and blow up M. The proper transform of $M \times \mathbb{C}$ meets the exceptional divisor – the normal bundle N plus a projectivized normal bundle at infinity – such that M is the zero section of the normal bundle. Let i be the inclusion of this zero section, $\mathcal{N}$ the conormal bundle, and π the projection. For any coherent sheaf $\mathcal{F}$ on M,

$$i_*\mathcal{F} = \pi^*\mathcal{F} \otimes \Lambda^{\cdot}\pi^*\mathcal{N}$$

in the Grothendieck group. The fundamental class of the zero section M in the exceptional divisor is $c_n(\pi^*N)$. Writing $x \cdot M = i_*i^*x$ for $x = ch(\pi^*\mathcal{F})t(\pi^*N)^{-1}$ gives

$$ch\ i_*(\mathcal{F}) = i_*(ch\mathcal{F}\frac{t(M)}{i^*t(\mathbb{P}^N)}).$$

By the projection formula

$$ch\ i_*\mathcal{F} = t(\mathbb{P}^N)^{-1}i_*(ch(\mathcal{F})t(M))$$

Moving along to a different level $M \times \{p\} \subset M \times \mathbb{C}$, the same rule holds for the inclusion of M in projective space.

For any $j \geq 0$, the dimension of $\Gamma\mathcal{O}_{P^N}(j)$ is the number of degree j monomials in $N+1$ variables, this equals

$$t(\mathbb{P}^N) \cdot ch\mathcal{O}(j),$$

meaning the degree of the discrete part of the product in the Chow ring; note $t(\mathbb{P}^N) = t(\mathcal{O}(1))^{N+1}$. Writing $i_*\mathcal{F}$ in the Grothendieck group as an integer linear combination of such positive $\mathcal{O}(j)$ – by using the principal parts sequence – gives

1. Theorem (Hirzebruch Riemann Roch) For sufficiently positive divisors D the dimension of $|D|$ is the degree of the discrete part of $(1 + D + D^2/2! + ...)t(M)$.