Short course about Lie groups

This is the outline of a short course about Lie groups. It includes discussions with Alex Suciu.

In the process of classifying them, people like Cartan decided that since each Lie group has a normal series with simple quotient groups, one should focus firstly on simple Lie groups.

Also, there are reasons for restricting attention to compact Lie groups. For example, any fiber bundle whose structural group is a Lie group also admits as a structural group a maximal compact subgroup of it.

Finally, it makes sense to restrict attention to connected Lie groups.

Thus, an interesting part of the theory is the theory of compact, connected, simple Lie groups. However, it would not be correct to think that this subcategory has any intrinsic meaning, for instance, there should be no reason for people who use Lie groups to think that they should particularly be interested in the compact, simple, connected ones. It is only our choice in understanding the theory to choose this particular type.

So let G be a compact, connected, simple Lie group (over the real numbers). All the maximal tori in G are conjugate, and the cosets of any particular maximal torus form an interesting manifold.

In fact, this manifold has a natural structure of a complex projective variety T. It is the variety of Borel subgroups of the complexified Lie algebra.

This is a great advertisement for the theory of complex projective varieties, that starting with any simple compact connected *real* Lie group, we obtain one as cosets of a torus. The dimension as a complex projective variety is half the dimension as a real manifold, and all the very interesting and restrictive structure of the theory of complex projective varieties is there. For example, starting with the group of rotations of three space about a point, we obtain in this way the Riemann sphere.

It is a familiar fact that rotations of three dimensional space about a point are the same as rotations of a sphere, and here we have realized[1] as a subgroup of the holomorphic automorphism group of a projective variety; as it happens the most elementary projective variety, the projective line itself.

As the next simplification, one decides that one is actually more interested in the full (connected) holomorphic automorphism group of this projective variety T. It is sometimes called a 'generalized flag variety.' And the group is nothing but the complexification of the original compact Lie group, if it had been viewed as a real algebraic group. It is a simple complex Lie group.

When one speaks of a 'rational representation' of such a group, one means a morphism to the automorphism group of a complex vector space. There is no indeterminacy allowed, and the term 'polynomial representation' is sometimes meant to refer to a representation which has some special property with respect to matrix entries, we shall not use either term but merely speak of a 'representation' to mean an action on a vector space coming from a morphism.

All the irreducible representations of the (connected) holomorphic automorphism group G of T arise in a manner that is reminiscent of thinking of wave functions. They arise by extending the group action to a line bundle, and considering the group action on the vector space of global sections. One way of making such a line bundle with an action is to notice that the stabilizer subgroup G_t of each point $t \in T$ is a Borel subgroup of G. Then for each choice of t and each finite dimensional representation V of G_t there is a unique vector bundle with extended action whose sections on an open subset $U \subset T$ whose inverse image in G is the subset W are the holomorphic functions $f : W \to V$ such that $f(gt) = gf(t)$ whenever $gt = t$.

[1]Actually, a lesson learned over many pages in chemistry.pdf, the coarse structure of spectral lines of atoms does not rely on an identification of the double cover of the rotation group with any subgroup of the automorphisms of the sphere, rather recognize a semidirect product of both, and the fine structure invariant spaces of solutions for a diagonal subgroup

The choice of t is inessential here, it does not affect the isomorphism type of the representation of course. In the case of one-dimensional representations of G_t, these are merely one dimensional representations of the semisimple quotient group, the maximal torus of G. Via the Chern class map, we get a function from the character group of the maximal torus to $H^2(T, \mathbb{Z})$ and this can be proved to be an isomorphism. Moreover it can be shown that the Chow ring of T is discrete, it is isomorphic to the cohomology algebra.

What this means then is that every abstract line bundle with extended G action comes from a unique representation of the maximal torus, and that forgetting the G action never makes two non-isomorphic line bundles become isomorphic.

At this point one sees four theories nicely coinciding: the (what is called 'rational') representation theory of Lie groups, the theory of the integer cohomology ring of T, and the theory of algebraic cycles (Chow ring) of T

In the case of rotations of three space, recall that we decided we are more interested in the group G of holomorphic automorphisms of T, now the projective line. Then one and the same infinite group can be viewed as $H^2(T, \mathbb{Z})$, the equivalence algebraic cycles on the projective line (the Picard group), and the isomorphism types of finite dimensional irreducible representations of G.

If we knew that the integer cohomology algebra of T were generated in degree two, we would have expressions for all the cohomology classes as polynomials in the one dimensional representations of the maximal torus of G.

Althoough T is called a 'generalized flag variety,' it really only generalizes the saturated flag varieties; yet it maps onto any non-saturated flag variety. If we let B be a Borel subgroup of G, then there are finitely many parabolic subgroups P of G containing B, and $T = G/B$ maps onto G/P for each choice of P. Thus in the case of special linear groups, T maps onto the Grassmannian varieties, and the cohomology classes of Grassmannian varieties, which are known as the universal Chern classes, then are contained in the cohomology algebra of T.

Analagous to the way people sometimes use the 'splitting principle' in the theory of characteristic classes, it is very useful to choose a better variety T' mapping onto T so that $H^2(T, \mathbb{Z}) \to H^2(T', \mathbb{Z})$ is an isomorphism, while the cohomology algebra of T' really is generated in degree two. Then this does allow us to express any element not only in the Chow ring of T but also in case of the special linear groups, any universal Chern class, as a polynomial in the one dimensional representations of the maximal torus.

That is, the theory of 'Chern roots' is sometimes interpreted as only a virtual theory, not an actual theory. Yet, we can embed the cohomology algebra of T into a larger ring that really is generated in lowest nontrivial degree, and such calculations are not virtual at all, they can be visualized as actual intersections of algebraic cycles.

The construction of T' is called by Demazure the 'Bott Samelson' construction, and I'll explain what this is.

Before leaving our discussion of T, it makes sense to pause a minute and mention however the notion of the 'root system' which often occurs in discussions of simple groups. This is the 'dual fan' of a toric variety (a compact projective variety with T action) naturally associated to G, and also it makes sense to relate the maps $T = G/B \to G/P$ to blowing down in a larger variety. We'll return to these later.

Let's now leave the discussion of T and go to the discussion of the better variety T'.

It is possible to represent the Lie algebra of G by matrices such that the direct sum of upper triangular, lower triangular, and diagonal matrices induces a decomposition of the Lie algebra of G into parts $N = N^+, N^-, H$, and note that $H + N^+$ is the Lie algebra of a Borel subgroup $B = B^+$ while $H + N^-$ is the Lie algebra of another Borel subgroup B^-. So choose $t \in T$ so that $B^- = G_t$. Then the point t is a fixed point for the vector fields belonging to N^- and H.

The quotient of N modulo its radical decomposes into one dimensional representations of the maximal torus fixing t (the one whose Lie algebra is H). These are sometimes indexed by the simple reflections, elements of the normalizer of that torus modulo the torus which transform each representation to its inverse.

If one denotes these basic reflections $s_1, ..., s_r$ then one can encode by words in the free group generated by the s_i the result of exponentiating the corresponding vector fields in a particular order and taking the closure.

Thus in SL_3 one denotes by s_1 the projective line which results by exponentiating t using the vector field belonging to the one dimensional representation sent to its inverse under s_1, and $s_1 s_2 s_1$ denotes the result of exponentiating in this direction some (complex) amount of time, then in the s_2 direction some amount of time, then again in the s_1 direction.

There are two ways of thinking about this. Just as when we exponentiate a simple Lie algebra we can get either a simply connected group or the same modulo any discrete central subgroup, here, when we exponentiate, we can choose either to ignore, or not to ignore, coincidences.

In the case of SL_3 what I am referring to is the fact that if I exponentiate for *zero* time in the s_2 direction, in the variety described by $s_1 s_2 s_1$, then I am describing the subvariety $s_1 s_1$. But exponentiating in the s_1 direction, stopping, and exponentiating some more, describes only upon passing to the closure the projective line itself.

I can choose to work formally and view $s_1 s_1$ as describing a copy of $\mathbb{P}^1 \times \mathbb{P}^1$, or I can insist that I want the two factors identified.

What this describes for SL_3, is a map from a three dimensional iterated $\mathbb{P}^1$ bundle to the three dimensional flag variety, the set of lines and planes in three space subject to the incidence relation of inclusion, defined by the single equation $x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$ in $\mathbb{P}^2 \times \mathbb{P}^2$.

That is, it is possible to blow up a projective line in that variety to obtain a three dimensional iterated $\mathbb{P}^1$ bundle.

We define T' by choosing any maximal length word in the Weyl group (a word of maximal length in the free group whose image in the Weyl group has the same length) and doing this construction.

We could also perform blowups of T to obtain T'. A warning is that the blowup while it can be done by blowing up a sheaf of ideals, the subscheme of T defined by the ideals would be allowed to be larger than the locus of indeterminacy of the inverse rational map. For example if I blow up a projective line in T to a copy of $\mathbb{P}^1 \times \mathbb{P}^1$ this may be done by blowing up a subscheme, but this subscheme will have its inverse image being a divisor, and therefore will have associated reduced subscheme larger than $\mathbb{P}^1$.

The underlying abelian group of the cohomology algebra of T' is just an exterior algebra on $H^2(T, \mathbb{Z})$, and the multiplication law is totally determined by the Cartan matrix of G together with the iterated bundle structure.

In fact, the role of the Weyl reflections enters in a transparent and natural way in its relation with the iterated bundle structure.

If we think that the projective line is the one point compactification of the line which we obtain by exponentiating t in one of the basic directions, it is simplest to exponentiate 'all the way to infinity' and to take the point at infinity for the starting point of the next exponentiation. Then the Weyl reflection arises when we change perspective and consider the point at infinity as the new starting point. That is, there is a chain of projective lines in T with a marked basepoint at the intersection of two of the lines. As we move our attention along the chain of projective lines, step-by-step the change of perspective interchanges what we thought of as the point zero and the point infinity of the next and the next projective lines.